SUMMIT MATH

Learn at your **OWN** pace.

ADVANCED ALGEBRA & TRIGONOMETRY SERIES

BOOK 6

UNIT 11: GRAPHING TRIGONOMETRIC FUNCTIONS

UNIT 12: PERMUTATIONS, COMBINATIONS & PROBABILITY

INTRODUCTION

Learning math through Guided Discovery:
Guided Discovery is designed to help you experience a sense of discovery as you learn each topic.

Why this curriculum series is named Summit Math:
Learning through Guided Discovery is like hiking. Learning and hiking both require effort and persistence and people naturally move at different paces, but they can reach the summit if they keep moving forward. Whether you race rapidly through the book or step slowly through each scenario, this textbook is designed to keep advancing your learning until you reach the summit.

Guided Discovery Scenarios:
The Guided Discovery Scenarios in this book are written and arranged to show you that new math concepts are related to previous concepts you already know. Try each scenario on your own first, check your answer when you finish, and then fix any mistakes, if needed. Making mistakes and struggling are essential parts of the learning process.

Homework & Extra Practice Scenarios:
After you complete the scenarios in each Guided Discovery section, extra practice will help you develop better memory of each topic. Use the Homework & Extra Practice Scenarios to improve your understanding and to increase your retention.

The Answer Key:
The Answer Key is included to give you teacher-like guidance. When you finish a scenario, you can get immediate feedback. If the Answer Key does not help you fully understand the scenario, try to get additional guidance from another student or a teacher.

Find more resources at:
www.summitmath.com

SAMPLE PAGES

GUIDED DISCOVERY SCENARIOS

The Guided Discovery Scenarios are designed to help you experience a sense of discovery as you learn. Explanations are brief and carefully timed to allow you to build your learning at a comfortable pace. The Answer Key supports your learning by giving you immediate feedback and helping you understand each step before moving on.

As you complete each scenario, follow these steps.

Step 1: Try the scenario.
Read the scenario carefully and try to work through it. Be willing to struggle.

Step 2: Check the Answer Key.
The Answer Key can guide you through the scenario, show you new ideas, or help you find mistakes in your steps.

Step 3: Fix your mistakes, if needed.
Mistakes are learning opportunities. If your result does not match the Answer Key, check your work and look for errors. If you still need guidance, seek help from a classmate or teacher.

When you are ready, try the next scenario and repeat this cycle.

CONTENTS

Unit 11

Graphing Trigonometric Functions

Section 1

Sine & Cosine Transformations

Use this page for taking notes or anything else that helps you learn.

Sine & Cosine Functions

1. Evaluate each ratio.

   a. sin(0°) b. sin(90°) c. sin(180°) d. sin(270°) e. sin(360°)

2. Evaluate each ratio.

   a. cos(0°) b. cos(90°) c. cos(180°) d. cos(270°) e. cos(360°)

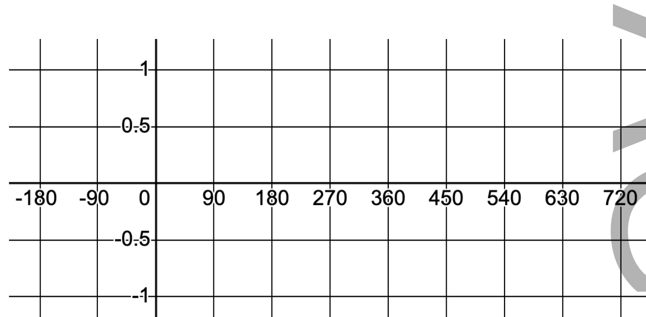
3. As you move around the Unit Circle in either positive or negative directions, the sine and cosine ratios follow repeating patterns. Identify the sine ratios for the angles shown in the table.

θ	−180°	−90°	0°	90°	180°	270°	360°	450°	540°
sinθ									

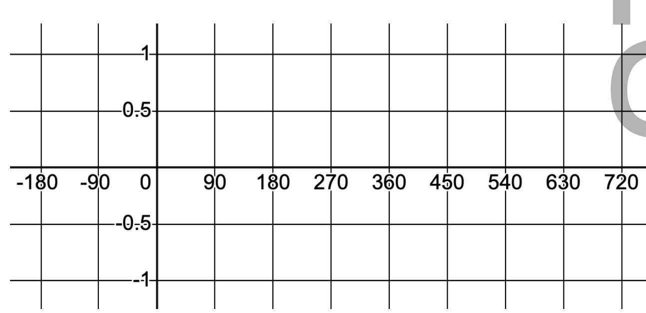
4. Identify the cosine ratios for the angles shown.

θ	−180°	−90°	0°	90°	180°	270°	360°	450°	540°
cosθ									

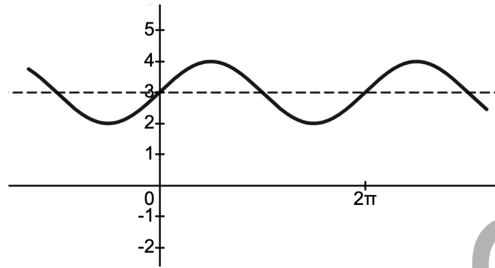
5. When you plot the sine and cosine ratios on a Cartesian Plane, they form waves. In the graph shown, draw as much of the sine wave as you can fit. First, mark the "key points" (where the sine ratio is 0, 1 or −1). Then, draw a wave through the points. The *x*-axis is in degrees.



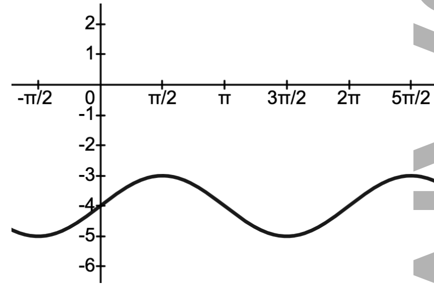
6. Draw as much of the cosine wave as you can fit in the graph. First, mark the "key points" (where the cosine ratio is 0, 1 or −1). Then, draw a wave through the points. The *x*-axis is in degrees.



12. Sine and cosine waves reach minimum and maximum values repeatedly. Halfway between these extremes is the "middle" of the wave, the midline. You can draw the midline with a dashed line, as shown in the graph. What is the equation of this curve's midline?



13. Consider the sine function shown.



a. Using a dashed line, draw the wave's midline.

b. The equation of this curve is $y = \sin\theta - $ ____.

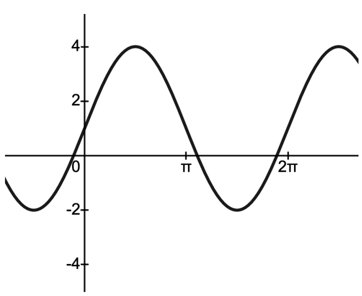
14. Identify the equation of the midline for each function shown.

a. $y = \sin\theta + 7$

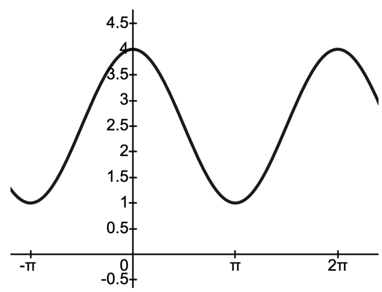
b. $y = 4\cos\theta$

15. A vertical shift can be represented by k. For each function below, use a dashed line to draw the wave's midline and then identify the k-value.

a. $y = 3\sin\theta + \mathrm{k}$



b. $y = 1.5\cos\theta + \mathrm{k}$



16. Sine and cosine waves have different starting points. One cycle of a sine wave starts on the midline while a cosine wave starts at a maximum. Write the coordinates of the starting point for one cycle of each function shown.

a. $y = \sin\theta$

(0, ____)

b. $y = \cos\theta$

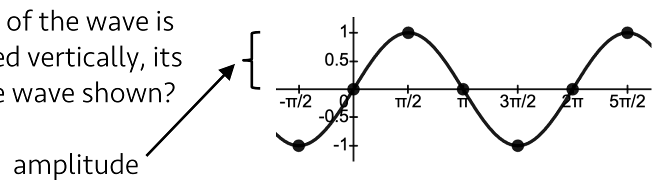
(0, ____)

c. $y = \sin\theta - 2$

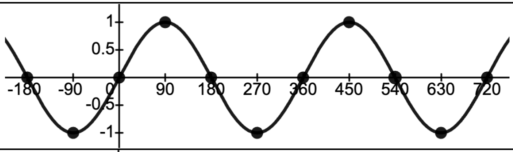
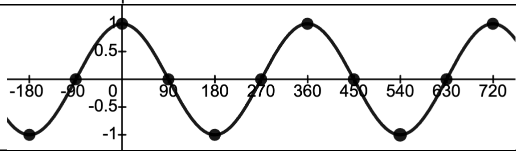
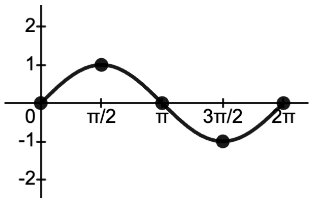
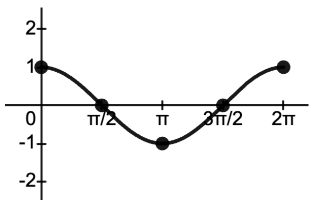
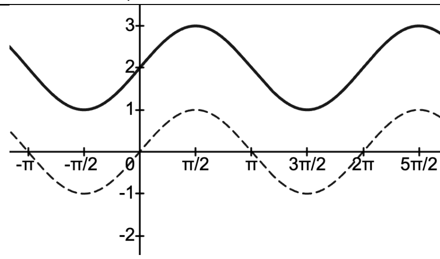
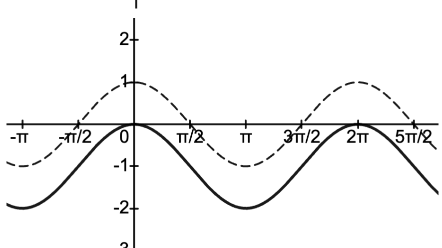
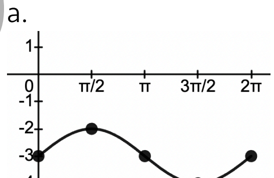
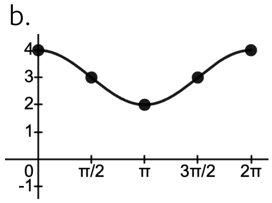
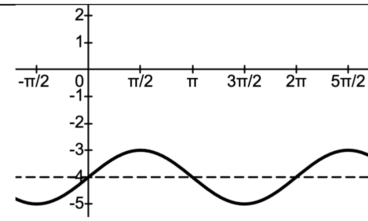
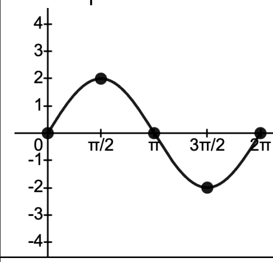
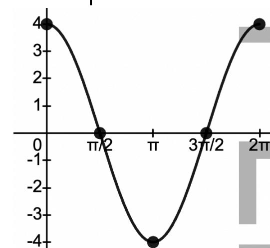
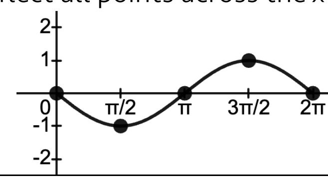
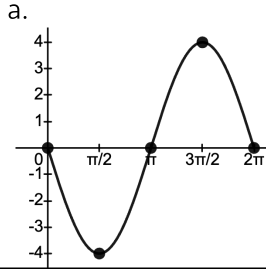
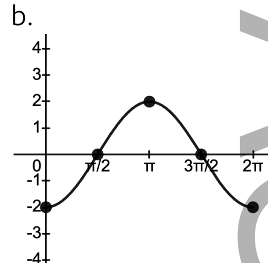
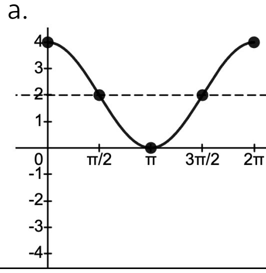
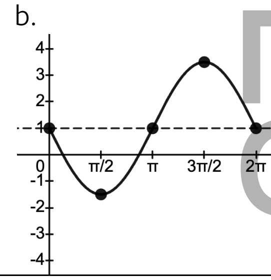
d. $y = \cos\theta + 3$

Vertical Stretches

17. The vertical distance from the midline to the top of the wave is the wave's amplitude. When the wave is stretched vertically, its amplitude changes. What is the amplitude of the wave shown?



Answer Key

1.	a. 0 b. 1 c. 0 d. −1 e. 0
2.	a. 1 b. 0 c. −1 d. 0 e. 1
3.	0 −1 0 1 0 −1 0 1 0
4.	−1 0 1 0 −1 0 1 0 −1
5.	
6.	
7.	a.  b. 
8.	a. $y = x^2 + 3$ b. $y = x^2 - 1$
9.	a. $y = \sin(x) - 3$ b. $y = \sin(x) + 1$ c. $y = \sin(x) - 2.8$
10.	a.  b. 
11.	a.  b. 
12.	$y = 3$
13.	a.  b. $\sin\theta - 4$
14.	a. $y = 7$ b. $y = 0$
15.	a. $k = 1$ b. $k = 2.5$
16.	a. $(0, 0)$ b. $(0, 1)$ c. $(0, -2)$ d. $(0, 4)$
17.	The amplitude is 1.
18.	a. 3 b. 6
19.	a. $y = 3\sin(x)$ b. $y = \frac{1}{4}\sin(x)$
20.	a. amplitude: 2  b. amplitude: 4 
21.	Reflect all points across the x-axis 
22.	a. 4 b. 2 c. 1.25 d. 1
23.	a.  b. 
24.	a. $A = 3$ b. $A = 1.5$
25.	a.  b. 
26.	a. $(0, 3)$ b. $(0, -4.5)$
27.	a. horizontal compression by a factor of 3 b. horizontal stretch by a factor of 5

Section 2

Graphing Sine & Cosine Functions

Use this page for taking notes or anything else that helps you learn.

Graphing Transformations of Sine & Cosine Functions

The next scenarios use what you have learned about transformations to graph sine and cosine functions in this form: $y = A\sin\big(B(x - h)\big) + k$ or $y = A\cos\big(B(x - h)\big) + k$. Focus on one transformation at a time, plot the 5 key points of one cycle, and then draw the wave.

1. The starting point for one cycle of $y = \sin(x)$ is $(0, 0)$. What is the starting point for each transformation shown?

   a. $y = \sin(x) + 3$
   b. $y = 4\sin\left(x - \frac{\pi}{2}\right)$
   c. $y = -2\sin\left(x + \frac{\pi}{3}\right) - 5$

2. The starting point for one cycle of $y = \cos(x)$ is $(0, 1)$. What is the starting point for each transformation shown?

   a. $y = \cos(x) - 4$
   b. $y = 5\cos\left(x + \frac{5\pi}{6}\right)$
   c. $y = -4\cos(x + 2) + 1$

3. Given the starting point and period for each sine function shown, identify the ending point.

   a. start: $\left(\frac{\pi}{6}, 2\right)$

      period: π

      end:

   b. start: $\left(-\frac{\pi}{4}, -1\right)$

      period: $\frac{3\pi}{4}$

      end:

4. Given the starting point and period for each cosine function shown, identify the ending point.

   a. start: $\left(\frac{3\pi}{2}, 5\right)$

      period: 4π

      end:

   b. start: $\left(-\frac{\pi}{3}, -3.5\right)$

      period: $\frac{5\pi}{2}$

      end:

5. Consider the function shown. Find the 5 key points of one cycle.

$$y = 3\sin\left(x - \frac{\pi}{6}\right) + 2$$

a. First, identify the midline and the amplitude.

midline: amplitude:

b. Second, identify the horizontal shift and use it to write the starting point of one cycle.

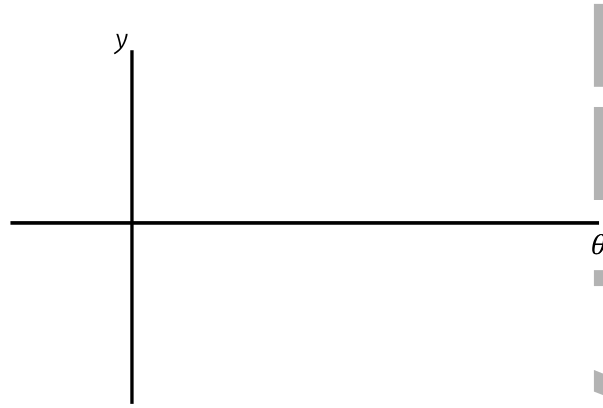
horizontal shift: starting point:

c. Third, identify the period and use it to find the ending point of one cycle.

period: ending point:

6. Graph one cycle of the function shown. Mark the axes to show the coordinates of the 5 key points.

$$y = 3\sin\left(x - \frac{\pi}{6}\right) + 2$$



7. Given the function, identify each item shown.

$$y = \cos(4x + \pi) - 3$$

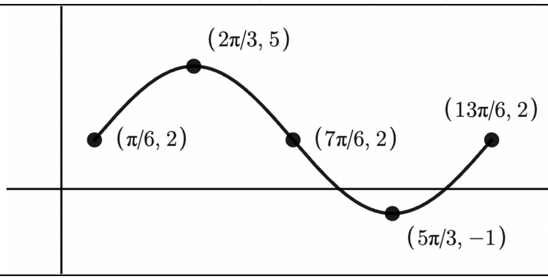
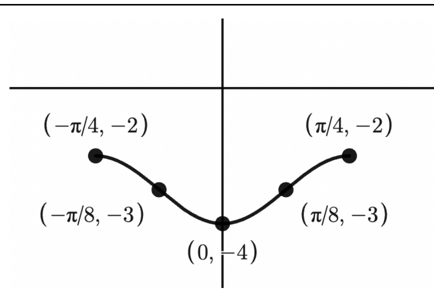
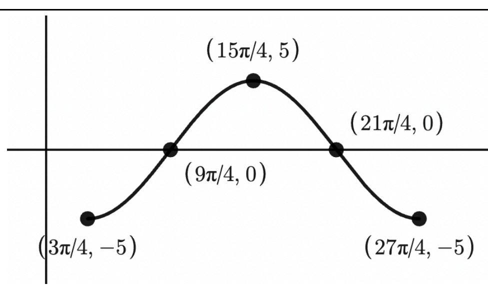
a. midline: b. amplitude:

c. horizontal shift: d. starting point:

e. period: f. ending point:

SAMPLE PAGES

Answer Key

1.	a. $(0, 3)$ b. $\left(\frac{\pi}{2}, 0\right)$ c. $\left(-\frac{\pi}{3}, -5\right)$
2.	a. $(0, -3)$ b. $\left(-\frac{5\pi}{6}, 5\right)$ c. $(-2, -3)$
3.	a. $\left(\frac{\pi}{6} + \pi, 2\right) \rightarrow \left(\frac{7\pi}{6}, 2\right)$ b. $\left(-\frac{\pi}{4} + \frac{3\pi}{4}, -1\right) \rightarrow \left(\frac{\pi}{2}, -1\right)$
4.	a. $\left(\frac{3\pi}{2} + 4\pi, 5\right) \rightarrow \left(\frac{11\pi}{2}, 5\right)$ b. $\left(-\frac{2\pi}{6} + \frac{15\pi}{6}, -3.5\right) \rightarrow \left(\frac{13\pi}{6}, -3.5\right)$
5.	a. midline: $y = 2$ amplitude: 3 b. shift: right $\frac{\pi}{6}$ start: $\left(\frac{\pi}{6}, 2\right)$ c. period: 2π end: $\left(\frac{\pi}{6} + 2\pi, 2\right) \rightarrow \left(\frac{13\pi}{6}, 2\right)$
6.	
7.	a. $y = -3$ b. 1 c. $\cos\left(4\left(x + \frac{\pi}{4}\right)\right) \rightarrow$ left $\frac{\pi}{4}$ d. $\left(-\frac{\pi}{4}, -2\right)$ e. $\frac{2\pi}{4} \rightarrow \frac{\pi}{2}$ f. end: $\left(-\frac{\pi}{4} + \frac{\pi}{2}, -2\right) \rightarrow \left(\frac{\pi}{4}, -2\right)$
8.	
9.	
10.	Two possible cosine equations: $y = 4\cos(3x) + 2$ $y = -4\cos(3(x - \pi)) + 2$
	Two possible sine equations: $y = 4\sin\left(3\left(x - \frac{\pi}{2}\right)\right) + 2$ $y = -4\sin\left(3\left(x - \frac{3\pi}{2}\right)\right) + 2$
11.	Two possible cosine equations: $y = 1.5\cos(2x) + 2$ $y = -1.5\cos\left(2\left(x - \frac{\pi}{2}\right)\right) + 2$ Two possible sine equations: $y = 1.5\sin\left(2\left(x - \frac{3\pi}{4}\right)\right) + 2$ $y = -1.5\sin\left(2\left(x - \frac{\pi}{4}\right)\right) + 2$
12.	a. The midline is 90, and the amplitude is 15, so the max. is 90 + 15 (105 mm of Hg) and the min. is 90 − 15 (75 mm of Hg). b. The period is $2\pi \div \frac{5\pi}{2} \rightarrow 2\pi \cdot \frac{2}{5\pi} \rightarrow \frac{4}{5}$. One heartbeat cycle is $\frac{4}{5}$ of a second. Every minute (60 seconds), their heart beats $60 \div \frac{4}{5} \rightarrow 60 \cdot \frac{5}{4} \rightarrow 75$ times. Their heartbeat is 75 beats per minute.
13.	a. The midline is 134.8, and the amplitude is 98.2, so the max. is 134.8 + 98.2 ($233,000) and the min. is 134.8 − 98.2 ($36,600). b. The average is the midline: $134,800
14.	$y = \mathrm{A}\sin(\mathrm{B}(x - \mathrm{h})) + \mathrm{k}$ $\mathrm{k} = 385{,}000$ $\mathrm{A} = 22{,}000$ period = 27.3 $27.3 = \frac{2\pi}{\mathrm{B}} \rightarrow \mathrm{B} = \frac{2\pi}{27.3}$ $D = 22{,}000\sin\left(\frac{2\pi}{27.3}t\right) + 385{,}000$
15.	The radius is 14 inches. The circumference is 28π inches, or 7.33 feet. The bike moves 20 feet per second, so the rock moves 7.33 feet (rotates around the tire once) in 0.3665 sec. Period = 0.3665 $\rightarrow \mathrm{B} = \frac{2\pi}{0.3665}$ $H = -14\cos\left(\frac{2\pi}{0.3665}t\right) + 14$ A cosine function is easier to use to model this because the rock starts on the ground, at its lowest height.

SAMPLE PAGES

Section 3

Graphing Reciprocal & Quotient Functions

Use this page for taking notes or anything else that helps you learn.

5. On the cosecant graph, there are <u>vertical asymptotes</u> where the cosecant ratio is undefined.

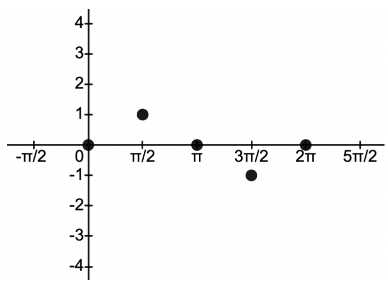
a. The cosecant function is undefined when the sine function is _____.

b. The function $y = \csc(x)$ is undefined when $x =$ ______, ______, ______, and so on...

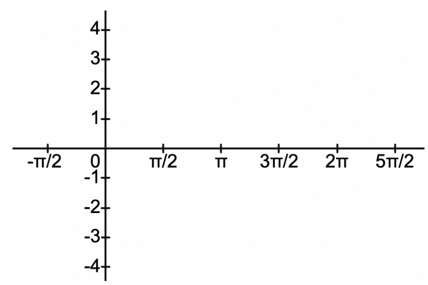
c. In the empty graph, use a dashed line to draw the first 3 vertical asymptotes of $y = \csc(x)$.

d. Plot the 2 points in the graph where the cosecant curve has a y-value of 1 or −1.

$y = \sin(x)$



$y = \csc(x)$



6. Now that you have 2 points and 3 vertical asymptotes, what does the rest of the cosecant curve look like? To determine the shape, think about the reciprocal of the sine values.

a. Simplify each fraction. Use a calculator, if needed.

$\frac{1}{0.2} =$ $\qquad$ $\frac{1}{0.1} =$ $\qquad$ $\frac{1}{0.01} =$ $\qquad$ $\frac{1}{0.001} =$

b. As the positive sine ratios approach 0, the cosecant ratios approach _____.

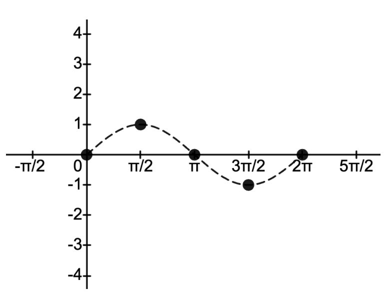
c. Simplify each fraction. Use a calculator, if needed.

$-\frac{1}{0.2} =$ $\qquad$ $-\frac{1}{0.1} =$ $\qquad$ $-\frac{1}{0.01} =$ $\qquad$ $-\frac{1}{0.001} =$

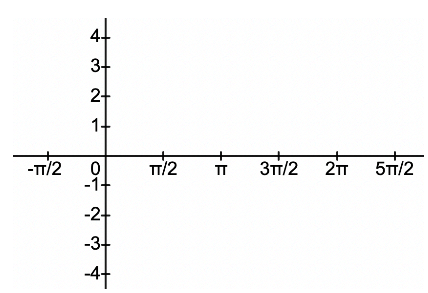
d. As the negative sine ratios approach 0, the cosecant ratios approach _____.

7. Use the sine function to draw one cycle of the cosecant curve over the domain $[0, 2\pi]$.

$y = \sin(x)$



$y = \csc(x)$



SAMPLE PAGES

The Tangent Function

The tangent ratio can be defined in several ways, as shown.

$\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$ $\qquad \tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\qquad \text{tangent} = \dfrac{\text{rise}}{\text{run}} = \text{slope}$

19. To build the graph of the tangent function, notice how a line's slope changes as its angle changes. Use "tangent = slope" to evaluate each expression. The angles are in radians.

a. $\tan(0)$ b. $\tan\left(\frac{\pi}{4}\right)$ c. $\tan\left(\frac{\pi}{2}\right)$ d. $\tan\left(\frac{3\pi}{4}\right)$ e. $\tan(\pi)$

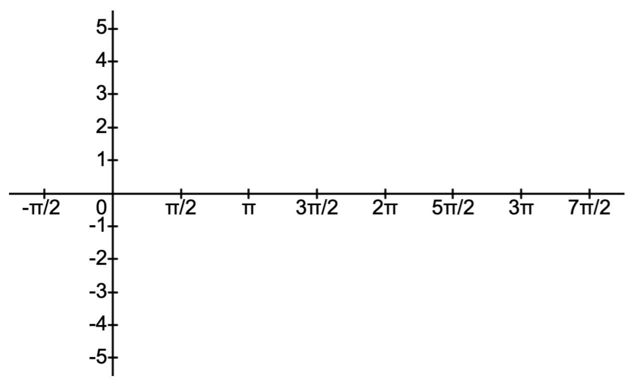
20. Evaluate the tangent ratio at each angle shown.

a. $-\pi$ b. $-\frac{\pi}{2}$ c. 0 d. $\frac{\pi}{2}$ e. π f. $\frac{3\pi}{2}$ g. 2π

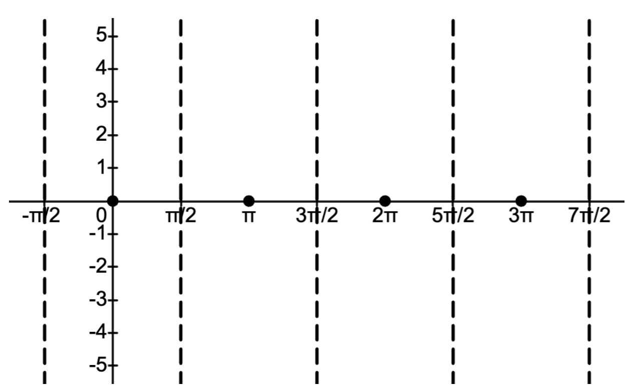
Like sine and cosine ratios, tangent ratios follow a repeating pattern. When an angle is horizontal, the tangent ratio is 0. When an angle is vertical, the tangent ratio is undefined.

When the tangent ratio is undefined, its graph has a vertical asymptote.
When the tangent ratio is 0, its graph has an *x*-intercept.

21. The next scenarios will help you graph the tangent function's repeating cycles. First, draw the vertical asymptotes: make a vertical dashed line at each angle where the tangent ratio is undefined. Next, plot the *x*-intercepts: draw a point at each angle where the tangent ratio is 0. In this graph, you can fit 5 asymptotes and 4 *x*-intercepts.



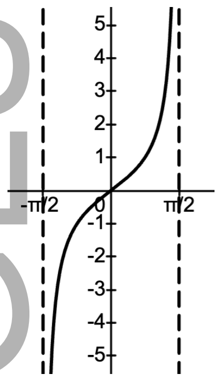
22. The asymptotes and *x*-intercepts do not provide enough information to visualize the shape of the tangent curve, but you can start to see its shape when you add the points where tangent is 1 or −1. In the graph shown, plot points at every angle where the tangent ratio is ±1.



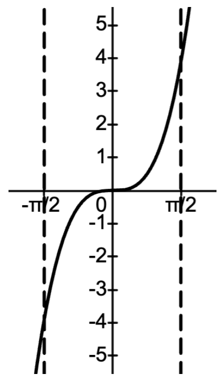
SAMPLE PAGES

23. You can use the points and asymptotes to draw the repeating tangent curves. Each tangent cycle looks like a cubic function, $y = x^3$. The first 2 graphs below show how 1 cycle of tangent is similar and also different than $y = x^3$. In the 3rd graph, draw 4 cycles of the tangent curve.

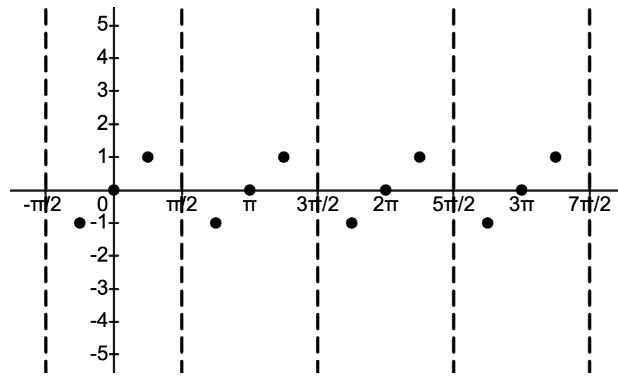
$y = \tan(x)$



$y = x^3$



$y = \tan(x)$



24. To better understand the shape of a tangent cycle, consider the first quadrant angles. In Figure 1 below, each angle is 15° apart. Each angle's slope is shown in Figure 2. At each point, the y-value is the angle's slope. The curve drawn through the points in Figure 2 is the graph of $y = \tan(x)$.

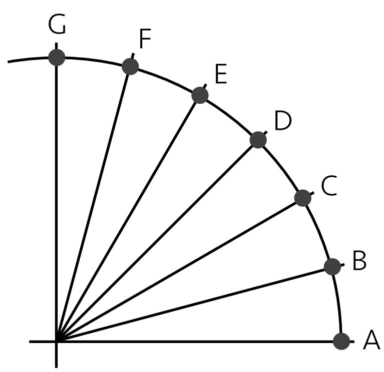


Figure 1

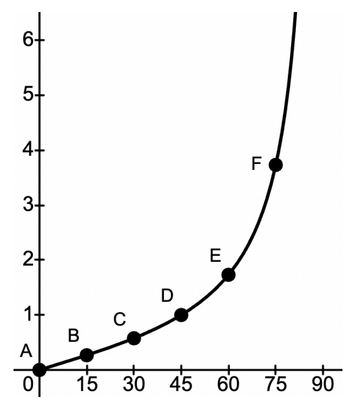


Figure 2

<u>4 tangent ratios are defined below, rounded to the nearest tenth:</u>
$\tan(15) \approx 0.3$ $\tan(30) \approx 0.6$ $\tan(60) \approx 1.7$ $\tan(75) \approx 3.7$

a. The 4 tangent ratios shown above correspond to angles in Figure 1. What is the slope of the segment that contains each of these points in Figure 1?

Point A: Point D: Point E: Point F:

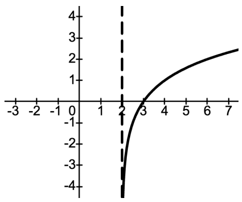
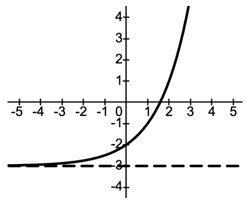
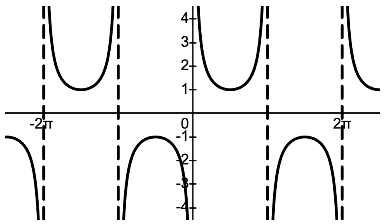
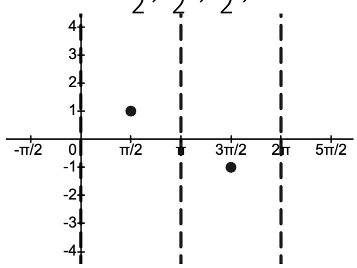
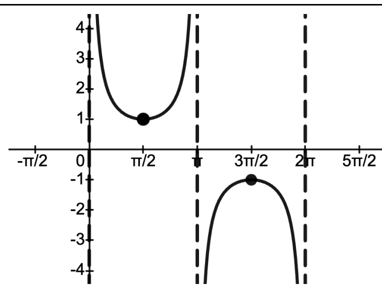
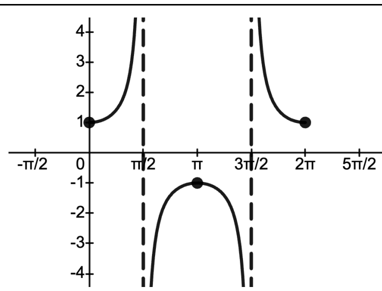
b. Fill in the blank for these points from Figure 2.

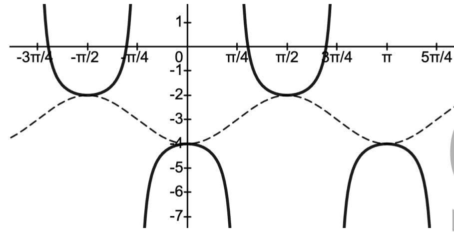
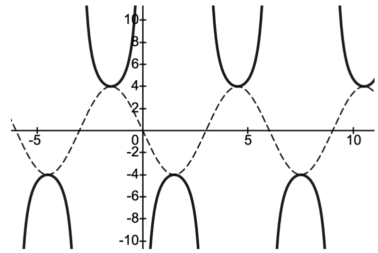
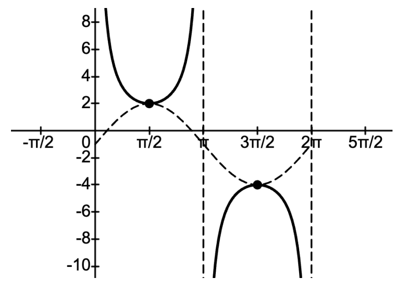
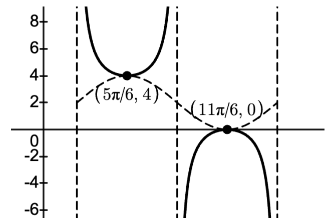
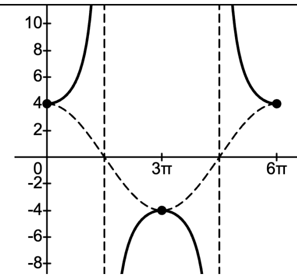
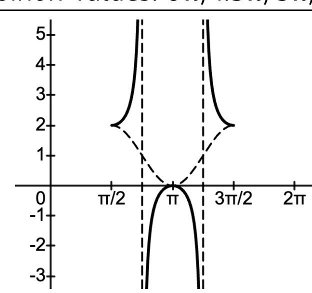
A: (0, ______) D: (45, ______) E: (60, ______) F: (75, ______)

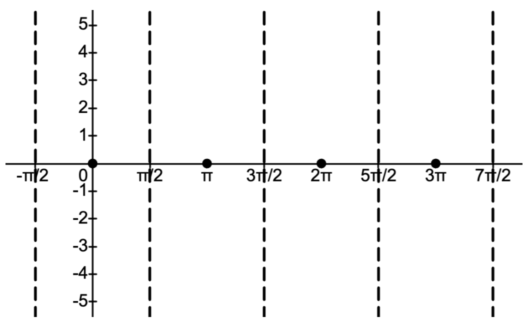
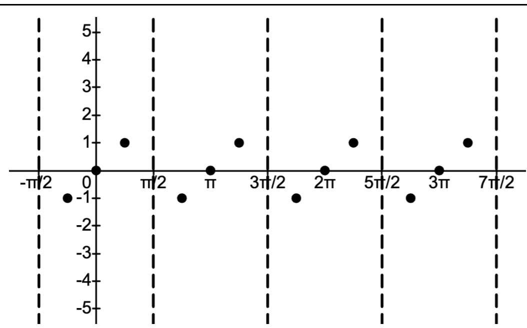
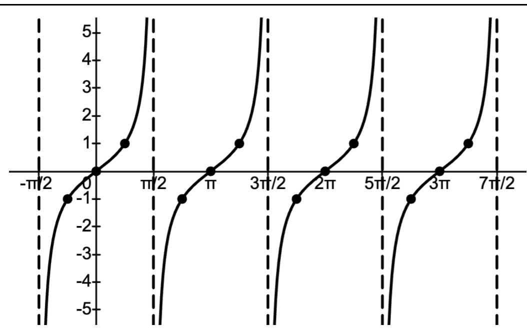
c. Why is point G not shown in Figure 2?

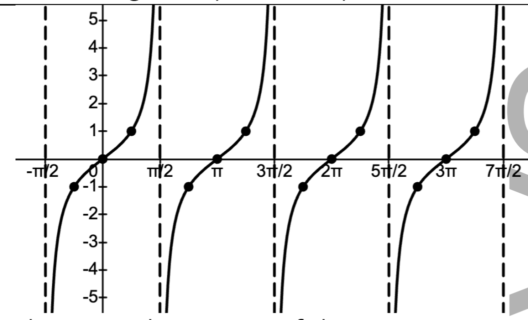
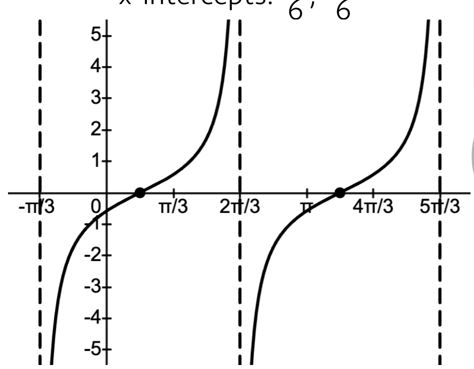
SAMPLE PAGES

Answer Key

1.	a. undefined b. 1 c. −1 d. 10 e. −100
2.	a. und. b. 1 c. und. d. −1 e. und.
3.	a.  b. 
4.	
5.	a. 0 b. $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$  c & d.
6.	a. $\frac{1}{0.2} = 5$ $\frac{1}{0.1} = 10$ $\frac{1}{0.01} = 100$ $\frac{1}{0.001} = 1000$ b. ∞ c. $-\frac{1}{0.2} = -5$ $-\frac{1}{0.1} = -10$ $-\frac{1}{0.01} = -100$ $-\frac{1}{0.001} = -1000$ d. $-\infty$
7.	
8.	

9.	a.  b. 
10.	cosine
11.	
12.	 5 key point x-values: $\frac{2\pi}{6}, \frac{5\pi}{6}, \frac{8\pi}{6}, \frac{11\pi}{6}, \frac{14\pi}{6}$
13.	 5 key point x-values: 0π, 1.5π, 3π, 4.5π, 6π
14.	 5 key point x-values: $\frac{2\pi}{4}, \frac{3\pi}{4}, \frac{4\pi}{4}, \frac{5\pi}{4}, \frac{6\pi}{4}$

15.	relative minimum
16.	vertical asymptote
17.	3 possible functions: $y = 2\sec\left(x - \frac{\pi}{3}\right) + 2$ $y = -2\sec\left(x - \frac{4\pi}{3}\right) + 2$ $y = -2\sec\left(x + \frac{2\pi}{3}\right) + 2$
18.	period: $6 \rightarrow b = \frac{\pi}{3}$ amplitude: 7 midline: -4 3 possible functions: 1. $y = -7\csc\left(\frac{\pi}{3}x\right) - 4$ 2. $y = -7\csc\left(\frac{\pi}{3}(x-3)\right) - 4$ 3. $y = -7\csc\left(\frac{\pi}{3}(x-6)\right) - 4$
19.	a. 0 b. 1 c. undefined d. -1 e. 0
20.	a. 0 b. undefined c. 0 d. und. e. 0 f. und. g. 0
21.	
22.	
23.	
24.	a. A: 0 D: 1 E: 1.7 F: 3.7 b. A: 0 D: 1 E: 1.7 F: 3.7 c. Point G is 90°, so its slope is undefined. Since tan(90°) is undefined, this is shown with a vertical asymptote in Figure 2. As the angles approach 90°, the slopes get steeper and approach ∞.

25.	a. Two angles with undefined tangent ratios: $\pm\frac{\pi}{2}$ b. $\pm\frac{\pi}{4}$
26.	a. 0 b. π: tangent repeats every 180°, or π rad.
27.	 The general equation of the x-intercepts is: $x = 0 + \pi n \rightarrow x = \pi n$
28.	a. $x - \frac{\pi}{2} \rightarrow$ shift the x-intercepts right $\frac{\pi}{2}$ $x = 0 + \pi n$ becomes $x = \frac{\pi}{2} + \pi n$ first 2 x-ints: $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{3\pi}{2}, 0\right)$ b. $\frac{1}{3}x \rightarrow$ multiply the x-intercepts by 3 $x = 0 + \pi n$ becomes $x = 0 + 3\pi n$ first 2 x-ints: (0, 0) and (3π, 0)
29.	$y = \tan\left(3\left(x - \frac{2\pi}{3}\right)\right)$ "h" is $-\frac{2\pi}{3} \rightarrow$ shift x-ints right $\frac{2\pi}{3}$ "b" is 3 $\rightarrow$ change the period to $\frac{\pi}{3}$ first 2 x-ints: $\left(\frac{2\pi}{3}, 0\right)$ and (π, 0)
30.	$x = \frac{\pi}{2}$ and $x = \frac{7\pi}{6}$ Asymptotes are halfway between x-ints. The average of $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ is $\frac{3\pi}{6}$, or $\frac{\pi}{2}$. The period is the distance between the intercepts, $\frac{4\pi}{6}$, so the second asymptote is $\frac{\pi}{2} + \frac{2\pi}{3}$, or $\frac{7\pi}{6}$.
31.	Shift the function right $\frac{\pi}{6}$. x-intercepts: $\frac{\pi}{6}$, $\frac{7\pi}{6}$ 

CONTENTS

Unit 12

Combinatorics: Counting & Arranging

Section 1

Permutations

Use this page for taking notes or anything else that helps you learn.

Counting Principles

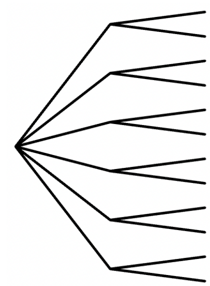
If you ever struggle to choose an outfit, this lesson may explain why. If you own 5 shirts and 3 pairs of pants, you can match each of the 5 shirts with each of the 3 pants to make 5 · 3 or 15 different outfits. If you have 4 pairs of shoes, the number of options multiplies to 15 · 4 or 60 unique outfits!

1. How many unique outfits can you make with 6 shirts, 4 pairs of pants and 3 pairs of shoes?

2. At the grocery store, there are 5 types of chips and 6 types of salsa. How many different combinations of chips and salsa can you make?

3. The previous scenario uses the Fundamental Counting Principle. If one event has P options and another event has Q options, then there are P · Q ways that both events can happen.

   Suppose you roll a 6-sided dice and then flip a coin. For example, one result is a 5 and tails. Another result is a 2 and heads. How many results are possible?



4. A tree diagram shows why the Fundamental Counting Principle involves multiplication. How does this tree diagram show the number of possible results in the previous scenario?

5. The total number of ways separate events can occur together is the product of the number of ways each event can occur separately.

   You can choose from 4 art classes, 5 languages and 3 writing classes. How many different schedules can you create if you choose 1 of each type of class?

6. A multiple-choice quiz has 10 questions, with four choices for each question (A, B, C, D). How many different ways can a student fill in the answers if they complete the quiz?

7. You want to guess which teams will win the first round of games in a 64-team tournament. There are 32 games in the first round. How many different ways can you predict who will advance to the second round of the tournament?

Given N Items, Arrange R

In the previous scenarios, all of the available items are being arranged. Now consider how many ways you can arrange items if you only choose some of them.

20. Suppose 10 students are in an art class, but only 2 of them will be chosen to present their final project tomorrow in class. Use the blanks below to help you calculate how many ways 2 can be chosen.

___ ___

a. How many students can be chosen to go first? Put this number in the first blank.

b. After the first student is chosen, how many students can be chosen to go second? Put this number in the second blank.

c. How many ways can 2 out of 10 students be chosen to present their project tomorrow?

21. If you have 6 books, how many ways can you arrange 3 of them on a shelf?

___ ___ ___

22. How many 4-digit codes can be made if every digit must be different and the digits can be 0 to 9?

___ ___ ___ ___

23. If 40 triathletes compete in the Olympics, how many ways can they finish 1st, 2nd and 3rd?

24. If you have n items in total and you arrange k of them, the permutation can be shown using the notation ${}_{n}P_{k}$. Find the value of each expression shown. Use a calculator to help you, if needed.

a. ${}_{6}P_{2}$

___ · ___

b. ${}_{10}P_{3}$

c. ${}_{30}P_{4}$

25. Another notation for permutations looks like the notation for functions: P(n, k). It is pronounced as "P of n comma k." Evaluate each expression shown.

a. P(9, 3)

b. P(12, 5)

SAMPLE PAGES

Answer Key

1.	$6 \cdot 4 \cdot 3 \rightarrow 72$ outfits
2.	$5 \cdot 6 \rightarrow 30$ combinations
3.	$6 \cdot 2 \rightarrow 12$ results
4.	The dice has 6 possible outcomes (branches) and the coin has 2 outcomes. The 6 dice branches (1 to 6) each have 2 coin branches (H or T). At the tips of the "tree", there are 6 sets of 2, or 12 possible outcomes.
5.	$4 \cdot 5 \cdot 3 \rightarrow 60$ schedule
6.	$4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$ $\rightarrow 4^{10} \rightarrow 1{,}048{,}576$ ways
7.	Each game has 2 possible outcomes. $2^{32} \rightarrow 4{,}294{,}967{,}296$ ways
8.	2 ways: AB BA
9.	6 ways: ABC ACB BAC BCA CAB CBA
10.	a. $4 \cdot 3 \cdot 2 \cdot 1 \rightarrow 24$ ways b. ABCD BACD CABD DABC ABDC BADC CADB DACB ACBD BCAD CBAD DBAC ACDB BCDA CBDA DBCA ADBC BDAC CDAB DCAB ADCB BDCA CDBA DCBA
11.	$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \rightarrow 120$ permutations
12.	a. $4 \cdot 3 \cdot 2 \cdot 1 \rightarrow 24$ b. $3 \cdot 2 \cdot 1 \rightarrow 6$ c. $2 \cdot 1 \rightarrow 2$
13.	$1! = 1$ $0! = 1$
14.	a. 720 b. 720 c. 720 d. $3 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \rightarrow 12$
15.	a. $6! = 720$ b. $7! = 5{,}040$
16.	a. $\frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5} = 4 \cdot 3 \cdot 2 \cdot 1 = 4!$ b. $\frac{4 \cdot 3 \cdot 2 \cdot 1}{4} = 3 \cdot 2 \cdot 1 = 3!$ c. $\frac{6 \cdot 5 \cdot 4 \cdot 3!}{3!} = 6 \cdot 5 \cdot 4 = 120$
17.	a. $\frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4!}{4! \cdot 3!} \rightarrow \frac{8 \cdot 7 \cdot 6 \cdot 5}{3!}$ $\rightarrow 8 \cdot 7 \cdot 5 \rightarrow 280$ b. $\frac{7 \cdot 6 \cdot 5!}{5! \cdot 2!} \rightarrow \frac{7 \cdot 6}{2!} \rightarrow 21$ c. $\frac{8 \cdot 7 \cdot 6!}{2! \cdot 6!} \rightarrow \frac{8 \cdot 7}{2!} \rightarrow 28$
18.	a. $n(n-1)(n-2)(n-3)\ldots$

	b. $(n-1)(n-2)(n-3)(n-4)\ldots$ c. $\frac{n(n-1)(n-2)\ldots}{n}$ $\rightarrow (n-1)(n-2)\ldots \rightarrow (n-1)!$
19.	a. $\frac{n(n-1)(n-2)(n-3)\ldots}{(n-1)(n-2)(n-3)\ldots} \rightarrow n$ b. $\frac{n(n-1)(n-2)(n-3)\ldots}{(n-2)(n-3)\ldots} \rightarrow n(n-1)$
20.	a & b. $\underline{10}$ $\underline{9}$ c. $10 \cdot 9 \rightarrow 90$
21.	$6 \cdot 5 \cdot 4 \rightarrow 120$
22.	$10 \cdot 9 \cdot 8 \cdot 7 \rightarrow 5{,}040$ codes
23.	$40 \cdot 39 \cdot 38 \rightarrow 59{,}280$ ways
24.	a. $6 \cdot 5 \rightarrow 30$ b. $10 \cdot 9 \cdot 8 \rightarrow 720$ c. $30 \cdot 29 \cdot 28 \cdot 27 \rightarrow 657{,}720$
25.	a. $9 \cdot 8 \cdot 7 \rightarrow 504$ b. $12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \rightarrow 95{,}040$
26.	a. 5 b. 8 c. 11 d. $n - k$
27.	a. 5 b. $8!$ c. $11!$ d. $\frac{n!}{(n-k)!}$
28.	a. $\frac{40!}{(40-3)!} \rightarrow \frac{40!}{37!} \rightarrow 59{,}280$ b. $\frac{25!}{(25-5)!} \rightarrow \frac{25!}{20!} \rightarrow 6{,}375{,}600$ c. $\frac{60!}{(60-11)!} \rightarrow \frac{60!}{49!} \rightarrow 1.37 \times 10^{19}$
29.	a. $\frac{8!}{3!} = \frac{n!}{(n-k)!}$ $n = 8$ $n - k = 3 \rightarrow 8 - k = 3 \rightarrow k = 5$ b. $\frac{13!}{6!} = \frac{n!}{(n-k)!}$ $n = 13$ $n - k = 6 \rightarrow 13 - k = 6 \rightarrow k = 7$
30.	a. $\frac{n!}{(n-1)!} \rightarrow \frac{n(n-1)!}{(n-1)!} \rightarrow n$ b. $\frac{n!}{(n-2)!} \rightarrow \frac{n(n-1)(n-2)!}{(n-2)!} \rightarrow n(n-1)$ c. $\frac{(n+1)!}{((n+1)-3)!} \rightarrow \frac{(n+1)!}{(n-2)!}$ $\rightarrow \frac{(n+1)(n)(n-1)(n-2)!}{(n-2)!}$ $\rightarrow n(n+1)(n-1) \rightarrow n(n^2-1) \rightarrow n^3 - n$

Section 2

Combinations

Use this page for taking notes or anything else that helps you learn.

Defining Combinations

If 3 students want to form a line, they can arrange their order 6 different ways. This is a permutation. If 3 students want to form a group of 3, how many ways can they do this? Since the order in which they select members of this group does not matter, they can only form one group of 3. In a permutation, order is important, but when the order does not matter, it is called a **combination**.

1. It can be hard to distinguish between permutations and combinations. Consider two examples.

    a. You can play 10 chords on a piano and you choose 5 of them to write a song. After you choose them, the order in which you play them matters. A different order creates a different song. Is this a permutation or combination?

    b. Eight pizza toppings are available, and you choose 3. After you mix them around on the top of a pizza, the order in which you put them on does not matter. A different order of the same toppings is the same flavor of pizza. Is this a permutation or combination?

Consider 2 examples of combinations.

You win a trip to Disney World and you get to bring 2 friends. You decide to pick Friend 1 and Friend 2. Then, you change your mind and pick Friend 2 and Friend 1. Did you change your trip? No. Even though the order is different, it is the same combination of friends.

Consider the letters Z, O, O. If one O is bold, **O**, there are 6 ways to arrange the 3 letters: Z**O**O, ZO**O**, **O**ZO, OZ**O**, **O**OZ, O**O**Z. There are 6 permutations. If no letter is bold, though, there are only 3 visibly different combinations of these letters: ZOO, OZO, OOZ.

2. To learn how to calculate combinations, start small and look for patterns. Suppose you are going camping overnight and you want to pack 2 shirts.

    a. If you have 5 shirts, how many ways can you pick 2?

    b. After you pick 2 shirts, you put them on your bed. How many ways can you arrange the 2 shirts on the bed?

3. There are $5 \cdot 4$ or 20 ways to select 2 out of 5 shirts: ${}_5P_2 = 20$. Since each pair of shirts can be arranged 2 ways, there are not 20 unique groups of 2. The number of different selections is half of 20. Since the order in which the shirts are chosen does not matter, it is a combination. The notation for this combination is ${}_5C_2$. The numerical value of ${}_5C_2$ is ____.

4. Try to use the previous scenario to calculate the value of ${}_5C_3$.

5. There are 7 hamsters in a pet store, and 3 are chosen by a family.

a. How many ways can 3 out of 7 hamsters be chosen?

b. The order in which you pick the 3 hamsters does not matter. They are all going to the same family. In how many different orders could those 3 hamsters be chosen?

6. There are 7 · 6 · 5 or 210 ways to select 3 out of 7 hamsters. Since each group of 3 can be chosen in 6 different ways, there are not 210 unique groups of 3. The number of different groups is 210 ÷ 6. Since the order of the selection does not matter, it is a combination. The value of ${}_7C_3$ is ____.

7. Try to use the previous scenario to calculate the value of ${}_7C_4$.

8. A combination is a permutation that is divided by the number of ways the selected items can be arranged. The formula for computing a combination is shown. Write this formula 2 more times.

$$ {}_nC_k = \frac{{}_nP_k}{k!} $$

9. Since ${}_nP_k = \dfrac{n!}{(n-k)!}$, the combination formula can also be written as shown. Write the simplified version of the formula 2 more times.

$$ {}_nC_k = \frac{\frac{n!}{(n-k)!}}{k!} \rightarrow \frac{n!}{(n-k)!} \cdot \frac{1}{k!} \rightarrow \frac{n!}{k!(n-k)!} $$

10. Calculate each combination using either of the 2 versions of the combination formula.

a. ${}_4C_2$

b. ${}_8C_3$

c. ${}_{11}C_5$

11. There are several notations that are used to express a combination. Three are shown below. Try to use a calculator to find the numerical value of each expression.

a. ${}_{10}C_3$

b. $C(16, 5)$

c. $\binom{72}{6}$

Answer Key

1.	a. permutation b. combination
2.	a. $5 \cdot 4 \rightarrow 20$ ways b. $2 \cdot 1 \rightarrow 2$ ways
3.	$20 \div 2 \rightarrow 10$
4.	$\frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1} \rightarrow \frac{60}{6} \rightarrow 10$
5.	a. $7 \cdot 6 \cdot 5 \rightarrow 210$ ways b. $3 \cdot 2 \cdot 1 \rightarrow 6$ ways
6.	$210 \div 6 \rightarrow 35$
7.	$\frac{7 \cdot 6 \cdot 5 \cdot 4}{4 \cdot 3 \cdot 2 \cdot 1} \rightarrow \frac{840}{24} \rightarrow 35$
8.	${}_nC_k = \frac{{}_nP_k}{k!}$ ${}_nC_k = \frac{{}_nP_k}{k!}$
9.	${}_nC_k = \frac{n!}{(n-k)!k!}$ ${}_nC_k = \frac{n!}{(n-k)!k!}$
10.	a. $\frac{{}_4P_2}{2!} = \frac{4 \cdot 3}{2!} = 6$ b. $\frac{{}_8P_3}{3!} = \frac{8 \cdot 7 \cdot 6}{3!} = 56$ c. $\frac{{}_{11}P_5}{5!} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{5!} = 462$
11.	a. 120 b. 4,368 c. 156,238,908
12.	a. comb. b. perm. c. perm. d. comb.
13.	${}_9C_3 = \frac{{}_9P_3}{3!} = \frac{9 \cdot 8 \cdot 7}{6} = 84$
14.	${}_{64}C_{24} = \frac{{}_{64}P_{24}}{24!} = 2.51 \times 10^{17}$
15.	${}_{1000}C_{80} = 5.43 \times 10^{119}$
16.	a. $\frac{{}_nP_1}{1!} = \frac{n}{1} = n$ b. $\frac{{}_nP_2}{2!} = \frac{n(n-1)}{2}$ c. $\frac{n(n-1)(n-2)...(3)(2)}{(n-1)!} = \frac{n(n-1)!}{(n-1)!} = n$
17.	a. $n = 8$ b. $\frac{n(n-1)}{2} = 21$ $\rightarrow n(n-1) = 42 \rightarrow n^2 - n = 42$ $\rightarrow n^2 - n - 42 = 0$ $\rightarrow (n-7)(n+6) = 0 \rightarrow n = 7$ or -6 n cannot be negative, so $n = 7$
18.	a. combination b. permutation c – e. combination
19.	$C(5, 2) \cdot C(7, 3) \cdot C(3, 1) = 10 \cdot 35 \cdot 3$ 1,050 ways
20.	$C(12, 3) \cdot C(16, 4) \cdot C(20, 5)$ $= 220 \cdot 1{,}820 \cdot 15{,}504 = 6{,}207{,}801{,}600$ ways
21.	$C(6, 4) \cdot C(2, 2)$ or $C(6, 2) \cdot C(4, 4)$

22.	$C(6, 4) = C(6, 2) = 15$ $C(2, 2) = C(4, 4) = 1$
23.	$C(10, 5) \cdot C(5, 3) \cdot C(2, 2) = 252 \cdot 10 \cdot 1$ or $C(10, 3) \cdot C(7, 2) \cdot C(5, 5) = 120 \cdot 21 \cdot 1$ or $C(10, 2) \cdot C(8, 5) \cdot C(3, 3) = 45 \cdot 56 \cdot 1$ 2,520 ways
24.	$C(8, 5) \cdot C(3, 3) = 56 \cdot 1$ or $C(8, 3) \cdot C(5, 5) = 56 \cdot 1$ 56 ways
25.	$C(15, 9) \cdot C(6, 4) \cdot C(2, 2) = 5005 \cdot 15 \cdot 1$ or $C(15, 4) \cdot C(11, 2) \cdot C(9, 9) = 1365 \cdot 55 \cdot 1$ or $C(15, 2) \cdot C(13, 9) \cdot C(4, 4) = 105 \cdot 715 \cdot 1$ 75,075 ways
26.	a. $8! \rightarrow 40{,}320$ b. $C(8, 2) \cdot C(6, 2) \cdot C(4, 2) \cdot C(2, 2)$ $= 28 \cdot 15 \cdot 6 \cdot 1 = 2{,}520$ ways c. $C(8, 3) \cdot C(5, 3) \cdot C(2, 2)$ $= 56 \cdot 10 \cdot 1 = 560$ ways
27.	a. 3! b. Arrange 3 letters (3!). Since 2 letters are repeated, divide by the number of ways the 2 M's can be arranged: $\frac{3!}{2!}$
28.	a. 4! b. Arrange 4 letters (4!). Since 2 letters are repeated, divide by the number of ways the 2 L's can be arranged: $\frac{4!}{2!} \rightarrow 12$
29.	a. 5! b. $\frac{5!}{2!} \rightarrow 60$ c. $\frac{5!}{2!2!} \rightarrow 30$
30.	a. $\frac{4!}{2!} \rightarrow 12$ b. $\frac{7!}{2!2!} \rightarrow 1{,}260$ c. $\frac{8!}{2!2!3!} \rightarrow 1{,}680$
31.	a. The first letter is R. 3 letters remain. They can be arranged 3!, or 6 ways b. Since the 2nd letter is R, there are 3 options for the 1st letter, 2 option for the 3rd, 1 option for the 4th. $3 \cdot 2 \cdot 1 \rightarrow 6$ ways
32.	N D _ _ _ 6 ways to fill 3 other blanks _ N D _ _ 6 ways to fill 3 other blanks _ _ N D _ 6 ways to fill 3 other blanks _ _ _ N D 6 ways to fill 3 other blanks 4 sets of 6 $\rightarrow$ 24 The order of ND can be switched to DN so there are 24 more ways: $24 \cdot 2 \rightarrow 48$ ways

Section 3
Probability

Use this page for taking notes or anything else that helps you learn.

Introduction to Probability

1. Suppose you own 8 shirts and 2 of them are blue. If you randomly select one, the probability that you choose a blue shirt is 2 out 8. As a fraction, the probability is $\frac{2}{8}$ or $\frac{1}{4}$. What is the probability that you do not choose a blue shirt?

The probability of an event is a ratio, defined as follows: $\text{probability} = \dfrac{\text{favorable outcomes}}{\text{possible outcomes}}$

2. Some probabilities are easier to calculate. Consider the two scenarios shown.

   a. A team of 10 players randomly selects 3 players to carry water to the game. What is the probability any one of the players is selected?

   b. 900 people apply for a scholarship and 20 are randomly selected to receive one. What is the probability a person does not win a scholarship?

3. In a standard deck of playing cards, there are 52 cards. There are four suits: hearts, diamonds, clubs, spades. There are 13 ranks (one of each suit) in the deck: ace, 2, 3, ..., 10, jack, queen, king. Hearts and diamonds are red. Clubs and spades are black. What is the probability of drawing...

   a. 1 queen?

   b. 1 spade?

4. What is the probability of drawing a card that is a queen AND a spade?

5. What is the probability of drawing a card that is either a queen OR a spade?

6. What is the probability of drawing a card that is...

   a. a 5 and red?

   b. a 5 or red?

7. Some probabilities require more computation, like ones that involve 2 events. Suppose you roll a 6-sided dice and then flip a coin. What is the probability of rolling a 5 or 6 and then landing on tails?

Step 1: Calculate "possible outcomes." There are 6 outcomes for each dice roll and then 2 outcomes for each coin flip. How many outcomes are possible?

Step 2: Calculate "favorable outcomes." One example of a favorable outcome is 5T: 5 followed by tails. How many outcomes are favorable?

Step 3: Write the probability as a fraction. Use a calculator to convert the result to a percent.

Probability of Independent Events

8. Look at the previous scenario a different way. It involves 2 independent events, rolling a dice and then flipping a coin. Their probabilities do not affect each other.

   a. What is the probability of rolling a 5 or 6?

   b. What is the probability of landing on tails?

   c. The probability of both events happening is their product. Multiply the two probabilities.

9. In a school, 60% of students play a sport, and 40% speak a second language fluently. What is the probability a student both plays a sport and speaks a second language fluently?

10. A survey of a local town reveals that 12% of the people are left-handed and 23% have a cat.

   a. What is the probability someone in the town is not left-handed and does not have a cat?

   b. What is the probability someone is left-handed or has a cat?

11. A basketball player makes 70% of their free throws, on average. At the end of the game, with their team losing by one point, they get to shoot 2 free throws. Are they likely to make both of them, if a "likely" event is defined as having a probability above 50%?

Answer Key

1.	$\frac{6}{8} \rightarrow \frac{3}{4}$
2.	a. $\frac{3}{10}$ b. $\frac{880}{900} \rightarrow \frac{44}{45}$
3.	a. $\frac{4}{52} \rightarrow \frac{1}{13}$ b. $\frac{13}{52} \rightarrow \frac{1}{4}$
4.	$\frac{4}{52} \cdot \frac{13}{52} \rightarrow \frac{1}{52}$
5.	Add the 4 queens and 13 spades. Subtract the 1 queen of spades counted twice. $\frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} \rightarrow \frac{4}{13}$
6.	a. There are two red 5's. $\frac{2}{52} \rightarrow \frac{1}{26}$ b. Add the four 5's and 26 reds. Subtract the two red 5's counted twice. $\frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{28}{52} \rightarrow \frac{7}{13}$
7.	Step 1: $6 \cdot 2 \rightarrow 12$ Step 2: 2 Step 3: $\frac{2}{12} \rightarrow \frac{1}{6} \approx 16.7\%$
8.	a. $\frac{2}{6} \rightarrow \frac{1}{3}$ b. $\frac{1}{2}$ c. $\frac{2}{6} \cdot \frac{1}{2} \rightarrow \frac{1}{6}$
9.	60% of 40% $\rightarrow 0.6 \cdot 0.4 \rightarrow 0.24 \rightarrow 24\%$
10.	a. 88% of 77% $\rightarrow 0.88 \cdot 0.77 \rightarrow 67.8\%$ b. Add the probability of each separate event and subtract the percent of the population that has been counted twice, those who are left-handed AND have a cat. $12\% + 23\% - 2.76\% = 32.24\%$

11.	70% of 70% $\rightarrow 0.7 \cdot 0.7 \rightarrow 0.49 \rightarrow 49\%$ It is less than 50%, so it is not likely.
12.	a. $\frac{25}{100} \cdot \frac{8}{50} \rightarrow \frac{1}{25} = 4\%$ b. You can expect to win 4% of 100 games, or 4 games, for a total of \$40.
13.	$\frac{13}{52} \cdot \frac{12}{51} \rightarrow \frac{1}{17} \approx 5.9\%$
14.	$\frac{26}{52} \cdot \frac{25}{51} \cdot \frac{24}{50} \rightarrow \frac{2}{17} \approx 11.8\%$
15.	$\frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \cdot \frac{10}{49} \cdot \frac{9}{48} \approx 0.05\%$
16.	The 1st digit must be a 5. There are 9 options for the 2nd digit, 8 options for the 3rd, and 7 options for the 4th. $\frac{1 \cdot 9 \cdot 8 \cdot 7}{10 \cdot 9 \cdot 8 \cdot 7} \rightarrow \frac{1}{10}$
17.	Step 1: 2^6 Step 2: ${}_6C_3$ Step 3: $\frac{{}_6C_3}{2^6} \approx 31.25\%$
18.	$\frac{{}_7C_2}{2^7} \approx 16.4\%$
19.	a. $\frac{{}_3C_3}{{}_7C_3} \approx 2.9\%$ b. $\frac{{}_3C_2 \cdot {}_4C_1}{{}_7C_3} \approx 34.3\%$
20.	$\frac{{}_2C_2 \cdot {}_8C_3}{{}_{10}C_5} \approx 22.2\%$
21.	a. ${}_{20}C_3 = 1{,}140$ b. $\frac{{}_3C_2 \cdot {}_{17}C_1}{{}_{20}C_3} \approx 4.5\%$

Section 4

Binomial Expansion

Use this page for taking notes or anything else that helps you learn.

Introduction

1. In a previous lesson, you learned how to count unique arrangements. How many distinguishable ways can the letters be arranged in the word MATH?

2. Some arrangements are not unique. For example, MOM and MOM are 2 different arrangements of the letters MMO, but they are not unique because they look the same in their current font and color. How many distinguishable ways can the letters be arranged in the word POPS?

3. Suppose the initial word is a string of letters instead: *xxyyy*. How many distinguishable ways can these letters be arranged?

There are 3 ways to think about the previous question, and each one produces the same result.

a. $\frac{5!}{2! \cdot 3!}$ → There are 5! ways to arrange 5 letters. Then, divide by the number of ways 2 identical *x*'s can be arranged (2!) and divide by the number of ways 3 *y*'s can be arranged (3!).

b. ${}_5C_2$ → In 5 blanks, you can arrange 2 *x*'s in ${}_5C_2$ distinguishable ways.

c. ${}_5C_3$ → In 5 blanks, you can arrange 3 *y*'s in ${}_5C_3$ unique ways.

4. How many distinguishable ways can the letters below be arranged? Use 2 different methods to get the same result.

aaaaabb

Raising a Binomial to an Exponent

The expression $(x + y)^N$ can be "expanded" and written as a polynomial. Two examples are shown.

$$(x + y)^2 = x^2 + 2xy + y^2 \qquad (x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

In this lesson, you will see how these binomial expansions relate to permutations and combinations.

5. When you expand the expression $(x + y)^2$ without using exponents or combining like terms, the expansion follows the steps below. Fill in the blank.

$(x + y)^2 \rightarrow (x + y)(x + y) \rightarrow xx + xy + yx +$ _____

14. Look at the expansion of $(x + y)^2$, which is $x^2 + 2xy + y^2$. The coefficients of the three terms are 1, 2, 1. The coefficients of $(x + y)^3$, or $x^3 + 3x^2y + 3xy^2 + y^3$, are 1, 3, 3, 1. There is also a relationship between a term's coefficient and its location in the expansion.

a. Each of the terms in $xx + xy + yx + yy$ contains 2 letters. In how many unique ways can you arrange the 2 letters in the term xy?

b. Each of the terms in $x^3 + 3x^2y + 3xy^2 + y^3$ contains 3 letters. In how many unique ways can you arrange the 3 letters in the term x^2y, or xxy?

15. Consider the expansion of $(x + y)^5$. How many ways can you create the term x^3y^2? Stated another way, how many unique ways can you arrange the 5 letters in the term $xxxyy$?

16. How many ways can you create the term x^4y^2?

17. Consider the expansion of $(x + y)^5$.

a. The first term is x^5: all 5 letters are x's. The number of unique ways 5 x's can be arranged in a group of 5 letters is ${}_5C_5$, and ${}_5C_5 =$ ______.

b. The second term of the expansion is x^4y: 4 of the letters are x. The number of ways 4 x's can be arranged in a group of 5 is ${}_5C_4$, and ${}_5C_4 =$ ______.

18. Use combination notation to find the coefficient for each term in the expansion. Notice how the exponents of x decrease while the exponents of y increase from left to right.

$$(x + y)^5 \rightarrow \_\_\_\_ x^5 + \_\_\_\_ x^4y + \_\_\_\_ x^3y^2 + \_\_\_\_ x^2y^3 + \_\_\_\_ xy^4 + \_\_\_\_ y^5$$

19. Use combination notation to fill in the missing coefficients.

$$(x + y)^6 \rightarrow \_\_\_\_ x^6 + \_\_\_\_ x^5y + \_\_\_\_ x^4y^2 + \_\_\_\_ x^3y^3 + \_\_\_\_ x^2y^4 + \_\_\_\_ xy^5 + \_\_\_\_ y^6$$

Answer Key

1.	$4! \rightarrow 4 \cdot 3 \cdot 2 \cdot 1 \rightarrow 24$
2.	$\frac{4!}{2!} \rightarrow 4 \cdot 3 \rightarrow 12$
3.	$\frac{5!}{3!2!} \rightarrow \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} \rightarrow 10$
4.	method 1: $\frac{7!}{5!2!} \rightarrow \frac{7 \cdot 6 \cdot 5!}{5! \cdot 2 \cdot 1} \rightarrow 21$ method 2: ${}_7C_5$ method 3: ${}_7C_2$
5.	yy
6.	This is a combination. ${}_3C_2$ or ${}_3C_1 \rightarrow 3$ There are 3 ways to arrange 2 x's or there are 3 ways to arrange 1 y.
7.	$xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$
8.	3: xxy, xyx, yxx ${}_3C_2 = 3$ ${}_3C_1 = 3$
9.	$yyyy$
10.	$1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$
11.	5 terms
12.	a. 3 b. 4 c. 6
13.	a. 9 b. $N + 1$
14.	a. 2 b. 3
15.	${}_5C_3$ or ${}_5C_2 \rightarrow 10$
16.	${}_6C_4$ or ${}_6C_2 \rightarrow 15$
17.	a. 1 b. 5
18.	$1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$ ${}_5C_5$ ${}_5C_4$ ${}_5C_3$ ${}_5C_2$ ${}_5C_1$ ${}_5C_0$
19.	${}_6C_6$ ${}_6C_5$ ${}_6C_4$ ${}_6C_3$ ${}_6C_2$ ${}_6C_1$ ${}_6C_0$ 1 6 15 20 15 6 1
20.	${}_7C_4$ or ${}_7C_3 \rightarrow 35$
21.	a. $7 + 2 \rightarrow 9$ b. ${}_9C_7$ or ${}_9C_2 \rightarrow 36$
22.	${}_6C_6(x)^6(y)^0 + {}_6C_5(x)^5(y)^1 + {}_6C_4(x)^4(y)^2$ $1 \cdot x^6 \cdot 1 + 6 \cdot x^5 \cdot y + 15 \cdot x^4 \cdot x^2$ $x^6 + 6x^5y + 15x^4y^2 + \ldots$

23.	${}_7C_7(x)^7(1)^0 + {}_7C_6(x)^6(1)^1 + {}_7C_5(x)^5(1)^2$ $1 \cdot x^7 \cdot 1 + 7 \cdot x^6 \cdot 1 + 21 \cdot x^5 \cdot 1$ $x^7 + 7x^6 + 21x^5 + \ldots$
24.	${}_8C_6(x)^6(2)^2$ or ${}_8C_2(x)^6(2)^2$ $28 \cdot x^6 \cdot 4 \rightarrow 112x^6$
25.	${}_9C_6(x)^6(-3)^3$ or ${}_9C_3(x)^6(-3)^3$ $84 \cdot x^6 \cdot -27 \rightarrow -2268x^6$
26.	${}_{10}C_4(y)^4(4)^6$ or ${}_{10}C_6(y)^4(4)^6$ $210 \cdot y^4 \cdot 4096 \rightarrow 860{,}160y^4$
27.	$(2x)^2 + 2(2x)(3y) + (3y)^2$ $\rightarrow 4x^2 + 12xy + 9y^2$
28.	$(3x)^2 + 2(3x)(-4y) + (-4y)^2$ $\rightarrow 9x^2 - 24xy + 16y^2$
29.	$(4x)^3 + 3(4x)^2(-y) + 3(4x)(-y)^2 + (-y)^3$ $\rightarrow 64x^3 - 48x^2y + 12xy^2 - y^3$
30.	a. 3 $(x^5 \ldots x^4 \ldots x^3)$ b. ${}_5C_3$ or ${}_5C_2 \rightarrow 10$ $\rightarrow 10(-2x)^3(3y)^2 \rightarrow 10(-8x^3)(9y^2)$ $\rightarrow -720x^3y^2$
31.	${}_6C_4(5x)^4(-2y)^2$ or ${}_6C_2(5x)^4(-2y)^2$ $15 \cdot 625x^4 \cdot 4y^2 \rightarrow 37{,}500x^4y^2$
32.	a. ${}_7C_2(2xy)^2(-7)^5$ or ${}_7C_5(2xy)^2(-7)^5$ $21 \cdot 4x^2y^2 \cdot -16807 \rightarrow -1{,}411{,}788x^2y^2$ b. ${}_9C_4(3x^2)^4(2y^3)^5$ or ${}_9C_5(3x^2)^4(2y^3)^5$ $126 \cdot 81x^8 \cdot 32y^{15} \rightarrow 326{,}592x^8y^{15}$
33.	${}_{10}C_4(4x^3)^4(-y)^6$ or ${}_{10}C_6(4x^3)^4(-y)^6$ $210 \cdot 256x^{12} \cdot y^6 \rightarrow 53{,}760x^{12}y^6$
34.	${}_7C_2(x^4)^2(-2y^3)^5$ or ${}_7C_5(x^4)^2(-2y^3)^5$ $21 \cdot x^8 \cdot -32y^{15} \rightarrow -672x^8y^{15}$