SUMMIT MATH

Learn at your **OWN** pace.

ADVANCED ALGEBRA & TRIGONOMETRY SERIES

BOOK 5

UNIT 9 : RADIANS, DEGREES & UNIT CIRCLE TRIGONOMETRY

UNIT 10 : TRIGONOMETRIC EQUATIONS & IDENTITIES

INTRODUCTION

Learning math through Guided Discovery:
Guided Discovery is designed to help you experience a sense of discovery as you learn each topic.

Why this curriculum series is named Summit Math:
Learning through Guided Discovery is like hiking. Learning and hiking both require effort and persistence and people naturally move at different paces, but they can reach the summit if they keep moving forward. Whether you race rapidly through the book or step slowly through each scenario, this textbook is designed to keep advancing your learning until you reach the summit.

Guided Discovery Scenarios:
The Guided Discovery Scenarios in this book are written and arranged to show you that new math concepts are related to previous concepts you already know. Try each scenario on your own first, check your answer when you finish, and then fix any mistakes, if needed. Making mistakes and struggling are essential parts of the learning process.

Homework & Extra Practice Scenarios:
After you complete the scenarios in each Guided Discovery section, extra practice will help you develop better memory of each topic. Use the Homework & Extra Practice Scenarios to improve your understanding and to increase your retention.

The Answer Key:
The Answer Key is included to give you teacher-like guidance. When you finish a scenario, you can get immediate feedback. If the Answer Key does not help you fully understand the scenario, try to get additional guidance from another student or a teacher.

Find more resources at:
www.summitmath.com

SAMPLE PAGES

GUIDED DISCOVERY SCENARIOS

The Guided Discovery Scenarios are designed to help you experience a sense of discovery as you learn. Explanations are brief and carefully timed to allow you to build your learning at a comfortable pace. The Answer Key supports your learning by giving you immediate feedback and helping you understand each step before moving on.

As you complete each scenario, follow these steps.

Step 1: Try the scenario.
Read the scenario carefully and try to work through it. Be willing to struggle.

Step 2: Check the Answer Key.
The Answer Key can guide you through the scenario, show you new ideas, or help you find mistakes in your steps.

Step 3: Fix your mistakes, if needed.
Mistakes are learning opportunities. If your result does not match the Answer Key, check your work and look for errors. If you still need guidance, seek help from a classmate or teacher.

When you are ready, try the next scenario and repeat this cycle.

CONTENTS

Unit 9

Trigonometry & The Unit Circle

SAMPLE PAGES

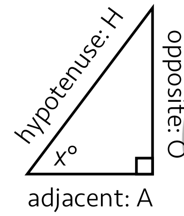
Section 1
Sine, Cosine & Tangent

Use this page for taking notes or anything else that helps you learn.

SAMPLE PAGES

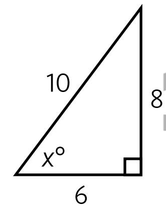
Reviewing Sine, Cosine & Tangent Ratios

1. In a previous lesson, you learned 3 ratios: sine, cosine and tangent. An acronym for remembering them is SOH CAH TOA. Use this acronym and the right triangle shown to define each ratio. Represent each word with its first letter.



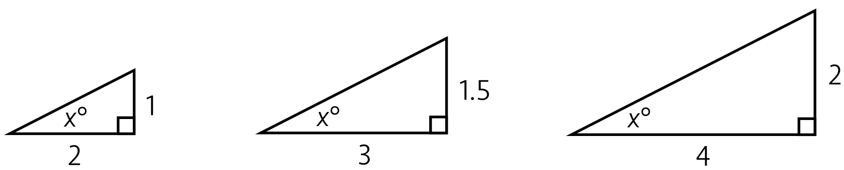
a. $\sin(x°) =$ b. $\cos(x°) =$ c. $\tan(x°) =$

2. Use the labeled triangle to identify each ratio.



a. $\sin(x°)$ b. $\cos(x°)$ c. $\tan(x°)$

3. In the previous scenario, the tangent ratio is $\frac{4}{3}$, but the side lengths are not 4 and 3. They are 8 and 6, which have a ratio of 4 to 3. Consider the triangles below. Each has a tangent ratio of 1 to 2 or $\frac{1}{2}$.



Since the triangles have the same side length ratio, they are similar triangles and they have the same angle measures. Estimate the measure of the angle labeled x.

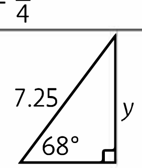
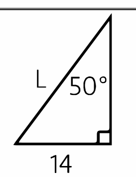
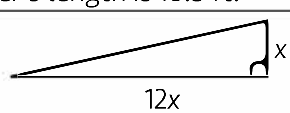
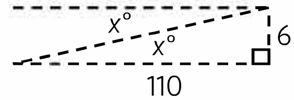
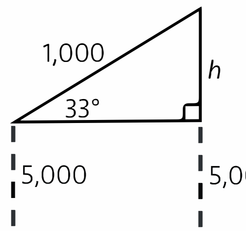
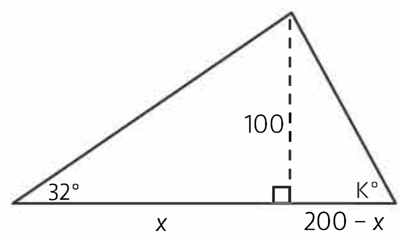
4. If you know an angle's trig ratio, you can use a calculator to find the size of the angle. If $\tan(x°) = 0.5$, then x is the "inverse tangent" of 0.5. Using more concise notation, $x = \tan^{-1}(0.5)$. Use a calculator to evaluate $\tan^{-1}(0.5)$ and the expressions below. Put the calculator in degree mode (not radian mode).

a. $\tan^{-1}(1)$ b. $\sin^{-1}(0.5)$ c. $\cos^{-1}(0.5)$

5. If you know one of the sides of a right triangle and one of the acute angles, you can solve a trig equation to find another side length. Use a calculator to solve each equation shown.

a. $\tan(41°) = \frac{x}{4}$ b. $\sin(0.5°) = \frac{8}{y}$

Answer Key

1.	a. $\sin(x°) = \frac{\text{opp.}}{\text{hyp.}}$ b. $\cos(x°) = \frac{\text{adj.}}{\text{hyp.}}$ b. $\tan(x°) = \frac{\text{opp.}}{\text{adj.}}$
2.	a. $\sin(x°) = \frac{8}{10} = \frac{4}{5}$ b. $\cos(x°) = \frac{6}{10} = \frac{3}{5}$ b. $\tan(x°) = \frac{8}{6} = \frac{4}{3}$
3.	Around 30°
4.	$\tan^{-1}(0.5) \approx 26.6°$ a. 45° b. 30° c. 60°
5.	a. $4\tan(41°) = x \rightarrow x \approx 3.48$ b. $y\sin(0.5°) = 8 \rightarrow y = \frac{8}{\sin(0.5°)} \approx 916.7$
6.	a. $x = \sin^{-1}(0.05) \approx 2.87°$ b. $x = \cos^{-1}\left(\frac{1}{3}\right) \approx 70.53°$
7.	a. $\tan(25°) = \frac{7}{x} \rightarrow x\tan(25°) = 7$ $\rightarrow x = \frac{7}{\tan(25°)} \approx 15.01$ b. $\cos(x°) = \frac{1.8}{3} \rightarrow x = \cos^{-1}\left(\frac{1.8}{3}\right) \approx 53.1°$
8.	a. $\tan(x°) = \frac{\text{opp.}}{\text{adj.}} \rightarrow \frac{\text{opp.}}{3} = \frac{4}{3} \rightarrow \text{opp.} = 4$ $3^2 + 4^2 = c^2 \rightarrow \text{hyp.} = 5$ $\sin(x°) = \frac{\text{opp.}}{\text{hyp.}} = \frac{4}{5}$ b. $\cos(y°) = \frac{\text{adj.}}{\text{hyp.}} \rightarrow \frac{\text{adj.}}{20} = \frac{4}{5} \rightarrow \text{adj.} = 16$ $16^2 + b^2 = 20^2 \rightarrow \text{opp.} = 12$ $\tan(y°) = \frac{\text{opp.}}{\text{adj.}} = \frac{12}{16} = \frac{3}{4}$
9.	Solve: $\sin 68° = \frac{y}{7.25}$ $\rightarrow y = 7.25\sin 68° =$ $\rightarrow y \approx 6.72$ ft The post's height is 6.7 ft. 
10.	Solve: $\sin 50° = \frac{14}{L}$ $\rightarrow L = \frac{14}{\sin(50°)}$ $\rightarrow L \approx 18.3$ ft The ladder's length is 18.3 ft. 
11.	 Solve $\tan x° = \frac{1}{12} \rightarrow x = \tan^{-1}\left(\frac{1}{12}\right) \approx 4.76°$ The ramp's angle of elevation is 4.8°.
12.	 The plane's angle of depression, x, equals the angle of elevation, x, looking up from the ground. Solve $\tan x° = \frac{6}{110} \rightarrow x = \tan^{-1}\left(\frac{6}{110}\right) \approx 3.1°$ The plane's angle of depression is 3.1°.
13.	 Solve: $\sin 33° = \frac{h}{1000} \rightarrow h = 1000\sin(33°)$ $\rightarrow h = 544.6$ ft $\rightarrow 544.6 + 5000 = 5{,}544.6$ ft The elevation at the top is 5,545 ft.
14.	 The height splits the triangle's base, 200, into 2 parts, "x" and "200 – x". $\tan 32° = \frac{100}{x} \rightarrow x = \frac{100}{\tan(32°)} \rightarrow x \approx 160$ To find Kyle's angle of elevation, solve: $\tan K° = \frac{100}{40} \rightarrow K = \tan^{-1}\left(\frac{100}{40}\right) = 68.2°$ Kyle's angle of elevation is 68.2°.
15.	$a = b\tan 20°$ $a = (b-4)\tan 45°$ $b\tan 20° = (b-4)\tan 45° \rightarrow b = 6.289$ $\rightarrow a = 2.289$
16.	$\tan 50° = \frac{y}{x}$ $\tan 70° = \frac{y}{x-30}$ $y = x\tan 50°$ $y = (x-30)\tan 70°$ $x\tan 50° = (x-30)\tan 70° \rightarrow x = 52.981$ $\rightarrow y = (52.981)\tan 50° \rightarrow 63.1$ feet The tower's height is 63.1 feet.

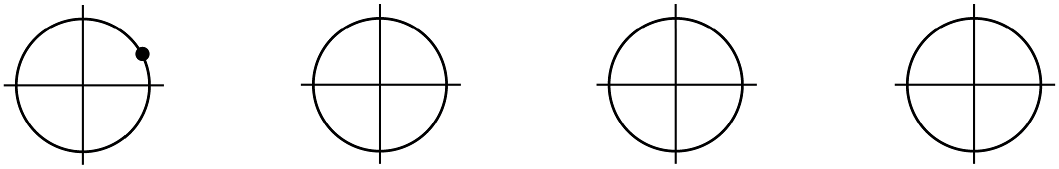
Section 2

Angles in Degrees & Radians

Use this page for taking notes or anything else that helps you learn.

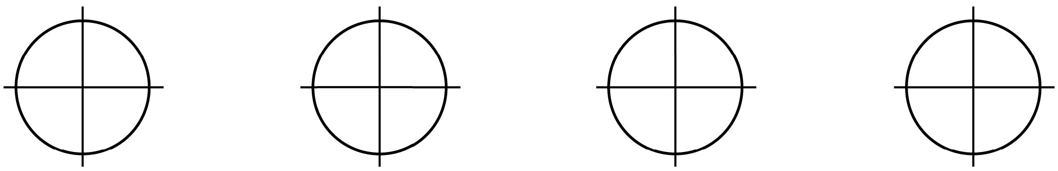
4. Use a point to mark the approximate location of each angle on the circle. The first is done for you.

a. 30°   b. 45°   c. 60°   d. 120°



5. Use a point to mark the approximate location of each angle on the circle.

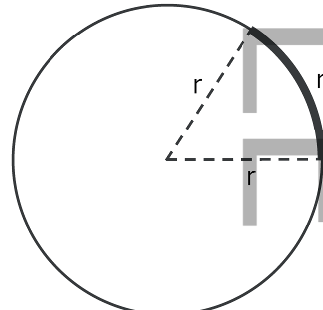
a. 210°   b. 270°   c. 135°   d. 330°



Angles in Radians

A circle's radius can also be used as a unit for measuring angles.

6. Start at 0° and move around the circle a distance of 1 radius length. This distance is highlighted on the circle shown and labeled with an r. How many radius lengths form a complete circle when connected end-to-end? Make a quick guess.



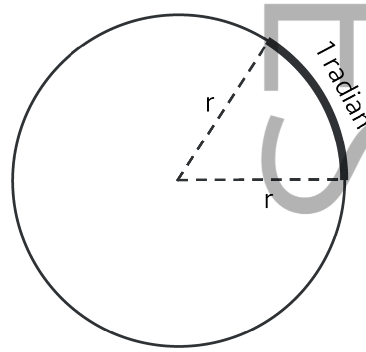
From now on, the phrase "radius length" is replaced by the word **radian**.

7. It takes slightly more than 6 **radians** to form a circle. What is the mathematical word for this length?

The distance around the outside of a circle is called the ____________________.

8. A circle's circumference is $2\pi r$ or 2π radians. The formula $C = 2\pi r$ shows that a circle's circumference is 2π radius lengths. What is the value of 2π, rounded to the nearest tenth?

9. The length represented by 1 radian is highlighted on the circle shown.



a. A circle is 360 degrees. A circle is also ________ radians.

b. How many radians form one-half of a circle?

10. What fractional amount of 180° is each angle shown?

a. 45° b. 30° c. 60°

$360° = 2\pi$ radians 180° is π radians 90° is $\frac{1}{2}$ of $\pi \rightarrow \frac{\pi}{2}$ radians

11. Convert each angle below to radians.

a. 45° b. 30° c. 60°

12. Write the 13 multiples of 30 from 0 to 360. The first 3 are already written. Continue the pattern.

0, 30, 60,

13. Now write the 13 multiples of $\frac{\pi}{6}$ from 0 to 2π.

a. The first 4 are already written. Continue the pattern, and do not simplify the fractions.

$\frac{0\pi}{6}, \frac{1\pi}{6}, \frac{2\pi}{6}, \frac{3\pi}{6},$

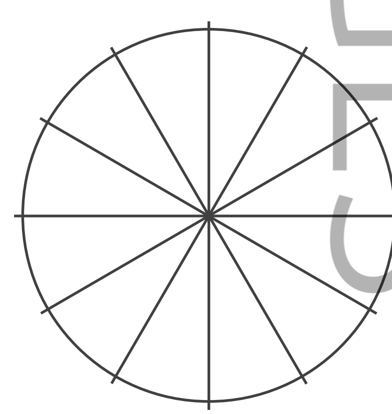
b. Rewrite the 13 multiples of $\frac{\pi}{6}$ from 0 to 2π. This time, simplify the fractions.

14. Write each angle below in radians.

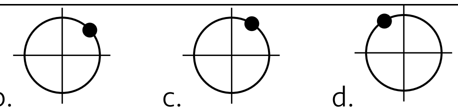
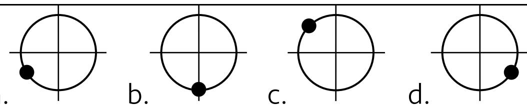
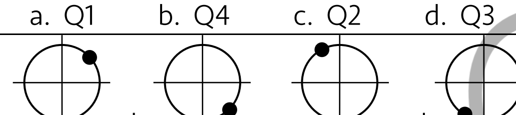
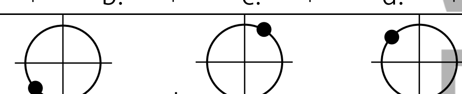
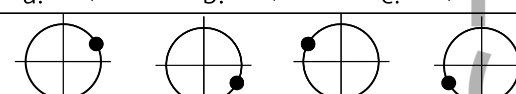
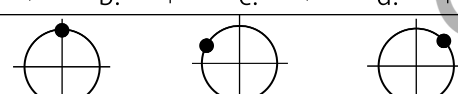
a. 90° b. 45° c. 30° d. 60°

15. The figure to the right shows a circle, split into equal sections.

At the tip of each segment, write the angle in radians.



Answer Key

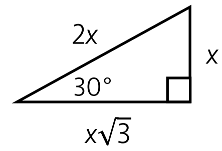
1.	a. 180 b. 90 c. 45 d. 30
2.	a. 30°, 60°, 90°, 120°, 150°, 180°, 210°, 240°, 270°, 300°, 330° b. 45°, 90°, 135°, 180°, 225°, 270°, 315°
3.	30°, 45°, 60°, 90°, 120°, 135°, 150°, 180°, 210°, 225°, 240°, 270°, 300°, 315°, 330°
4.	 b. c. d.
5.	 a. b. c. d.
6.	a little more than 6
7.	circumference
8.	6.28 → 6.3
9.	a. 2π radians b. $1\pi \rightarrow \pi$ radians
10.	a. $\frac{1}{4}$ b. $\frac{1}{6}$ c. $\frac{1}{3}$
11.	a. $\frac{1}{4}$ of $\pi \rightarrow \frac{\pi}{4}$ b. $\frac{1}{6}\pi \rightarrow \frac{\pi}{6}$ c. $\frac{\pi}{3}$
12.	0, 30, 60, 90, 120, 150, 180, 210, 240, 270, 300, 330, 360
13.	a. $\frac{0\pi}{6}, \frac{1\pi}{6}, \frac{2\pi}{6}, \frac{3\pi}{6}, \frac{4\pi}{6}, \frac{5\pi}{6}, \frac{6\pi}{6}, \frac{7\pi}{6}, \frac{8\pi}{6}, \frac{9\pi}{6}, \frac{10\pi}{6}, \frac{11\pi}{6}, \frac{12\pi}{6}$ b. $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{11\pi}{6}, 2\pi$
14.	a. $\frac{\pi}{2}$ b. $\frac{\pi}{4}$ c. $\frac{\pi}{6}$ d. $\frac{\pi}{3}$
15.	Write these angles: $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{11\pi}{6}$
16.	0, 45, 90, 135, 180, 225, 270, 315, 360
17.	a. $\frac{0\pi}{4}, \frac{1\pi}{4}, \frac{2\pi}{4}, \frac{3\pi}{4}, \frac{4\pi}{4}, \frac{5\pi}{4}, \frac{6\pi}{4}, \frac{7\pi}{4}, \frac{8\pi}{4}$ b. $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi$
18.	a. $120° \rightarrow 2 \cdot 60 \rightarrow 2 \cdot \frac{\pi}{3} \rightarrow \frac{2\pi}{3}$ b. $210° \rightarrow 7 \cdot 30 \rightarrow 7 \cdot \frac{\pi}{6} \rightarrow \frac{7\pi}{6}$ c. $300° \rightarrow 5 \cdot 60 \rightarrow 5 \cdot \frac{\pi}{3} \rightarrow \frac{5\pi}{3}$ d. $330° \rightarrow 11 \cdot 30 \rightarrow 11 \cdot \frac{\pi}{6} \rightarrow \frac{11\pi}{6}$
19.	a. $135° \rightarrow 3 \cdot 45 \rightarrow 3 \cdot \frac{\pi}{4} \rightarrow \frac{3\pi}{4}$ b. $225° \rightarrow 5 \cdot 45 \rightarrow 5 \cdot \frac{\pi}{4} \rightarrow \frac{5\pi}{4}$
	c. $315° \rightarrow 7 \cdot 45 \rightarrow 7 \cdot \frac{\pi}{4} \rightarrow \frac{7\pi}{4}$
20.	$2\pi \div 10 \rightarrow \frac{\pi}{5}$
21.	a. $\frac{20}{180} \rightarrow \frac{1}{9}$ (one-ninth) b. $\frac{18}{180} \rightarrow \frac{1}{10}$ c. $\frac{36}{180} \rightarrow \frac{1}{5}$ d. $\frac{1}{180}$
22.	a. $\frac{\pi}{90}$ b. $\frac{\pi}{60}$ c. $\frac{\pi}{2}$ d. $\frac{\pi}{3}$
23.	a. $20 \cdot 1° \rightarrow 20 \cdot \frac{\pi}{180} \rightarrow \frac{20\pi}{180} \rightarrow \frac{\pi}{9}$ radians b. $70 \cdot 1° \rightarrow 70 \cdot \frac{\pi}{180} \rightarrow \frac{70\pi}{180} \rightarrow \frac{7\pi}{18}$ radians c. $21 \cdot 1° \rightarrow 21 \cdot \frac{\pi}{180} \rightarrow \frac{21\pi}{180} \rightarrow \frac{7\pi}{60}$ radians
24.	a. $\frac{180°}{10} \rightarrow 18°$ b. $\frac{180°}{18} \rightarrow 10°$
25.	a. $\frac{3(180°)}{4} \rightarrow 3(45°) \rightarrow 135°$ b. $\frac{5(180°)}{6} \rightarrow 5(30°) \rightarrow 150°$ c. $\frac{7(180°)}{3} \rightarrow 7(60°) \rightarrow 420°$
26.	a. $\frac{360°}{\pi} \rightarrow 114.6°$ b. $\frac{540\pi°}{4\pi} \rightarrow 135°$
27.	a. 20° b. $\frac{\pi}{6} \cdot \frac{180°}{\pi} \rightarrow 30°$
28.	a. $\frac{5\pi}{6} \cdot \frac{180°}{\pi} \rightarrow 150°$ b. $\frac{1}{3} \cdot \frac{180°}{\pi} \rightarrow \frac{60°}{\pi} \rightarrow 19.1°$
29.	a. $\frac{\pi}{2}$ b. $\frac{\pi}{4}$ c. $\frac{\pi}{6}$ d. $\frac{\pi}{3}$
30.	a. $150° \rightarrow 5 \cdot 30° \rightarrow \frac{5\pi}{6}$ b. $240° \rightarrow \frac{8\pi}{6} \rightarrow \frac{4\pi}{3}$ c. $135° \rightarrow \frac{3\pi}{4}$ d. $330° \rightarrow \frac{11\pi}{6}$
31.	a. Q1 b. Q4 c. Q2 d. Q3
32.	 a. b. c. d.
33.	 a. b. c.
34.	 a. b. c. d.
35.	 a. b. c.
36.	a. $60 - 360 \rightarrow -300°$ b. $-210°$

Section 3

The Unit Circle

Use this page for taking notes or anything else that helps you learn.

5. After reviewing special right triangles, look at each type separately. Use the 30–60–90 triangle shown to identify these trig ratios.

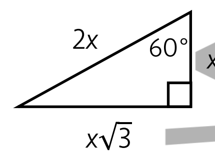


a. sin(30°) b. cos(30°)

6. To help you remember them, restate the simplified form of these 2 ratios.

a. sin(30°) b. cos(30°)

7. Now that you know two trig ratios for a 30° angle, consider a 60° angle. Use the previous triangle to identify each trig ratio shown.

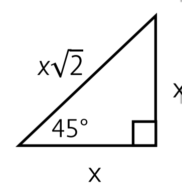


a. sin(60°) b. cos(60°)

8. To help you remember them, restate the simplified form of these 2 ratios.

a. sin(60°) b. cos(60°)

9. What are the ratios for a 45° angle? Use the 45–45–90 triangle to identify them.

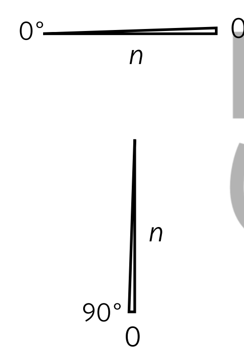


a. sin(45°) b. cos(45°)

10. To help you remember them, restate the simplified form of these 2 ratios.

a. sin(45°) b. cos(45°)

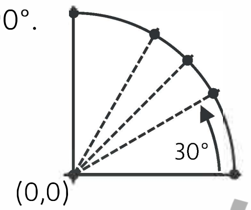
11. Now that you have used special right triangles to discover the sine and cosine ratios for 30°, 45° and 60°, consider two more angles, 0° and 90°. What does a right triangle look like if its base angle is 0°? What about 90°? Use the right triangles shown to help you evaluate these ratios.



a. sin(0°) b. sin(90°)

cos(0°) cos(90°)

12. You have now discovered sine and cosine ratios for common angles from 0° to 90°. In ascending order, the ratios are $0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1$. Try to recall the ratios shown.



a. $\sin(30°)$ b. $\cos(45°)$ c. $\sin(60°)$ d. $\cos(30°)$

It is easier to remember these ratios when you notice patterns. First, notice how the sine ratios can be written in an un-simplified form that shows a simple pattern.

	sin(0°)	sin(30°)	sin(45°)	sin(60°)	sin(90°)
simplified →	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
un-simplified →	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$

13. Try to recall the cosine ratios. Write them in both their un-simplified and simplified form.

$\cos(0°)$ $\cos(30°)$ $\cos(45°)$ $\cos(60°)$ $\cos(90°)$

14. The cosine ratios are also the sine ratios, but they match up with different angles. Fill in each blank to show which cosine ratio matches the given sine ratio.

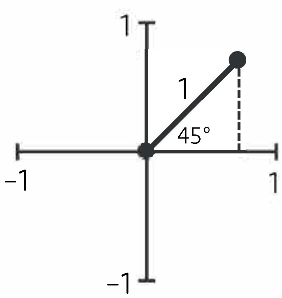
a. $\sin(30°) = \cos(\_\_\_\_\_)$ b. $\sin(60°) = \cos(\_\_\_\_\_)$ c. $\sin(0°) = \cos(\_\_\_\_\_)$

Building the Unit Circle

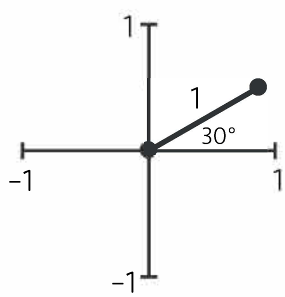
The ratios $0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1$ can be arranged on a circle with their corresponding angles. The next scenarios will show you where these ratios are on a Unit Circle, a circle with a radius of 1 "unit."

15. Each line segment shown has a length of 1 unit and an endpoint in Quadrant 1. Use special right triangles to find the coordinates of each endpoint.

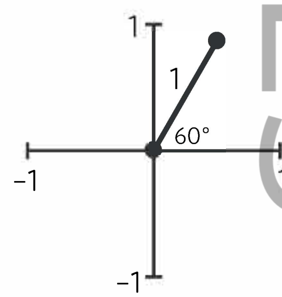
a.



b.

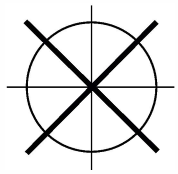


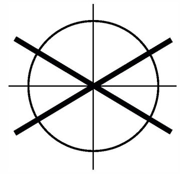
c.

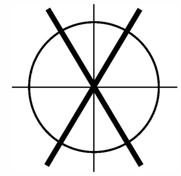


Grouping the Sine Ratios

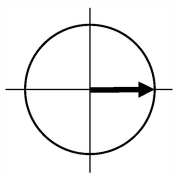
30. There are 16 angles on the Unit Circle but only 5 sine ratios: $0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1$. They are easier to recall when sorted into groups by reference angle. In each figure, 4 Unit Circle angles with the same reference angle are shown. What reference angle is represented in each figure?

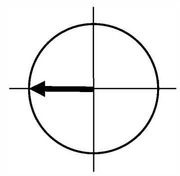
a.


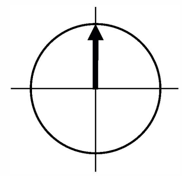
b.


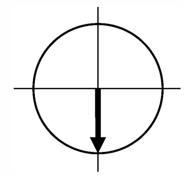
c.


31. First consider angles with a reference angle of either 0° or 90°. Using "sine is *y*," identify each ratio.

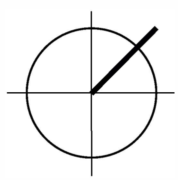
a. sin(0°)


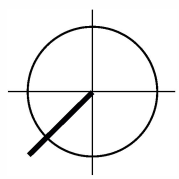
b. sin(180°)


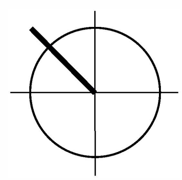
c. sin(90°)


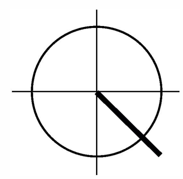
d. sin(270°)


32. Now consider angles with a reference angle of 45°. Identify each ratio.

a. sin(45°)


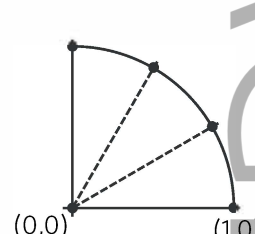
b. sin(225°)


c. sin(135°)


d. sin(315°)


33. Knowing an angle's sine ratio is its *y*-value on the Unit Circle can help you remember which ratio is $\frac{1}{2}$.

a. Which angle has a smaller *y*-value on the Unit Circle: 30° or 60°?

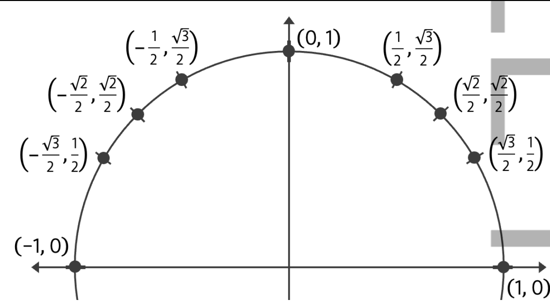
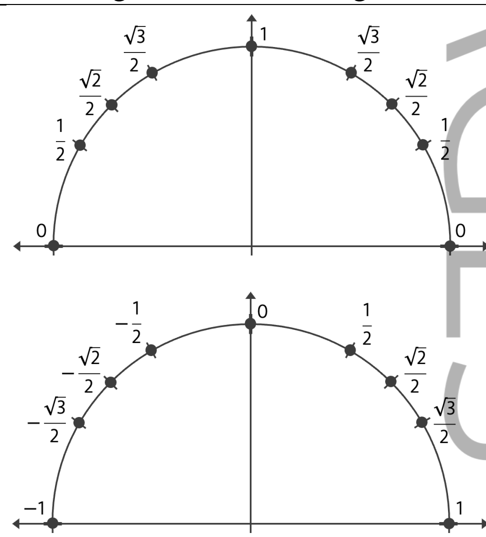


b. Which number is smaller: $\frac{1}{2}$ or $\frac{\sqrt{3}}{2}$?

c. Which ratio is $\frac{1}{2}$, sin(30°) or sin(60°)?

Answer Key

1.	a. $x\sqrt{3}$ b. $x\sqrt{2}$
2.	a. shorter leg: 1 longer leg: $\sqrt{3}$ b. longer leg: $5\sqrt{3}$ hypotenuse: 10 c. leg: 2 hypotenuse: $2\sqrt{2}$
3.	a. shorter leg: $2\sqrt{3} \rightarrow \frac{6}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{6\sqrt{3}}{3} = 2\sqrt{3}$ hypotenuse: $4\sqrt{3}$ b. both legs: $3\sqrt{2} \rightarrow \frac{6}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$
4.	a. shorter leg: $\frac{1}{2}$ longer leg: $\frac{1}{2} \cdot \sqrt{3} \rightarrow \frac{\sqrt{3}}{2}$ b. both legs: $\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
5.	a. $\frac{x}{2x} \rightarrow \frac{1}{2}$ b. $\frac{x\sqrt{3}}{2x} \rightarrow \frac{\sqrt{3}}{2}$
6.	a. $\sin(30°) = \frac{1}{2}$ b. $\cos(30°) = \frac{\sqrt{3}}{2}$
7.	a. $\sin(60°) = \frac{x\sqrt{3}}{2x} = \frac{\sqrt{3}}{2}$ b. $\cos(60°) = \frac{x}{2x} = \frac{1}{2}$
8.	a. $\sin(60°) = \frac{\sqrt{3}}{2}$ b. $\cos(60°) = \frac{1}{2}$
9.	$\sin(45°) = \cos(45°) = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
10.	$\sin(45°) = \cos(45°) = \frac{\sqrt{2}}{2}$
11.	a. $\sin(0°) = \frac{0}{n} = 0$ The adjacent side equals the hypotenuse so $\cos(0°) = \frac{n}{n} = 1$. b. The opposite side equals the hypotenuse, so $\sin(90°) = \frac{n}{n} = 1$ $\cos(90°) = \frac{0}{n} = 0$
12.	a. $\sin(30°) = \frac{1}{2}$ b. $\cos(45°) = \frac{\sqrt{2}}{2}$ c. $\sin(60°) = \frac{\sqrt{3}}{2}$ d. $\cos(30°) = \frac{1}{2}$
13.	1 $\frac{\sqrt{3}}{2}$ $\frac{\sqrt{2}}{2}$ $\frac{1}{2}$ 0
14.	a. $\sin(30°) = \cos(60°)$ b. $\sin(60°) = \cos(30°)$ c. $\sin(0°) = \cos(90°)$
15.	a. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ b. $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ c. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ 30-60-90 triangle → Since the hypotenuse is 1, the legs are $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$. 45-45-90 triangle → hypotenuse: 1, legs: $\frac{\sqrt{2}}{2}$

16.	a. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ b. $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ c. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
17.	30°: $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ 45°: $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ 60°: $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ 0°: (1,0) 90°: (0,1)
18.	a. $\frac{0.8}{1} \rightarrow 0.8$ b. $\frac{0.6}{1} \rightarrow 0.6$ c. (0.8, 0.6)
19.	a. $\frac{x}{1} \rightarrow x$ b. $\frac{y}{1} \rightarrow y$ c. (x, y)
20.	a. $\cos(30°) = \frac{\sqrt{3}}{2}$ b. $\sin(30°) = \frac{1}{2}$
21.	a. $\sin(0°) = 0$ $\sin(30°) = \frac{1}{2}$ $\sin(45°) = \frac{\sqrt{2}}{2}$ $\sin(60°) = \frac{\sqrt{3}}{2}$ $\sin(90°) = 1$ b. $\cos(0°) = 1$ $\cos(30°) = \frac{\sqrt{3}}{2}$ $\cos(45°) = \frac{\sqrt{2}}{2}$ $\cos(60°) = \frac{1}{2}$ $\cos(90°) = 0$
22.	a. quadrants 2 & 3 b. quadrants 3 & 4
23.	angles between 90° and 270° (Q2 & Q3)
24.	
25.	a. Q1, 2 b. Q1, 4
26.	a. Q2 b. Q3 c. Q3 d. Q2 e. Q3
27.	a. negative b. negative
28.	a. b. 

Section 4

Tangent & Reciprocal Ratios

Use this page for taking notes or anything else that helps you learn.

Common Tangent Ratios

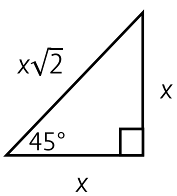
1. In previous scenarios, the variable x has been used to represent an angle. For example, if $\sin(x) = 1$, then x is 90°. Another common variable used for angles is θ. The symbol θ is spelled "theta" and pronounced THAY-tuh or THAY-duh). Determine the value of θ, in degrees, for each ratio shown.

   a. If $\cos(\theta) = \frac{1}{2}$, then $\theta =$______.

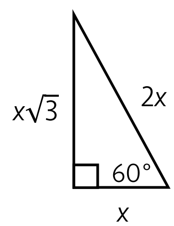
   b. If $\sin(\theta) = -\frac{1}{2}$, then $\theta =$______.

2. You can find the tangent ratio for a given angle using the equation $\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$. Find the tangent ratio for the marked angle in each triangle shown.

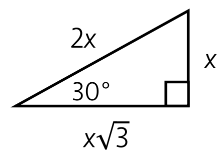
   a.



   b.



   c.

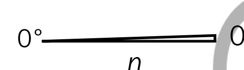


These are the tangent ratios for the 3 common angles in Quadrant 1.

$\tan(30°) = \frac{\sqrt{3}}{3}$ $\tan(45°) = 1$ $\tan(60°) = \sqrt{3}$

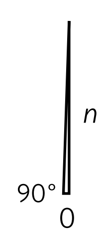
3. Now consider the tangent ratios for 0° and 90°.

   a. A right triangle with a 0° angle has an opposite side length of 0 so its tangent ratio is $\frac{0}{n}$. What is the value of the fraction $\frac{0}{n}$ if $n \neq 0$?



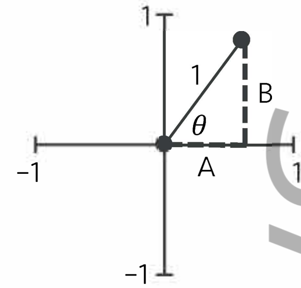
   $\tan(0°) =$

   b. A right triangle with a 90° angle has an adjacent side length of 0 so the tangent ratio is $\frac{n}{0}$. What is the value of the fraction $\frac{n}{0}$ if $n \neq 0$?



   $\tan(90°) =$

4. It is easier to remember tangent ratios if you connect them to something familiar. To help you with this, consider the figure shown.

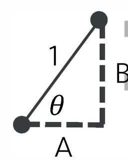


a. Identify each ratio.

$\cos(\theta)$ $\sin(\theta)$ $\tan(\theta)$

b. Given the values above, $\tan(\theta) =$

5. Consider the previous diagram again.



a. The slope of the hypotenuse is ____ and $\tan(\theta) =$ ____.

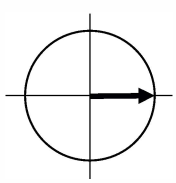
b. What can you conclude about the tangent ratio?

tangent is ___________

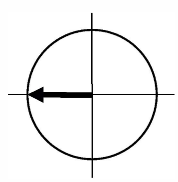
An angle's tangent ratio is the slope of the angle's terminal side. On the Unit Circle, there are 5 tangent ratios (5 slopes): 0, $\frac{\sqrt{3}}{3}$, 1, $\sqrt{3}$, undefined. You can sort them by reference angle.

6. First consider angles with a reference angle of either 0° or 90°. A horizontal line's slope is ____ and a vertical line's slope is __________________. Use "tangent is slope" to identify each ratio.

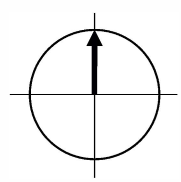
a. $\tan(0°)$



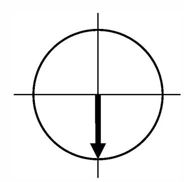
b. $\tan(180°)$



c. $\tan(90°)$

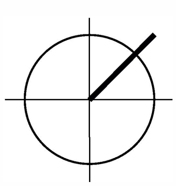


d. $\tan\left(\frac{3\pi}{2}\right)$

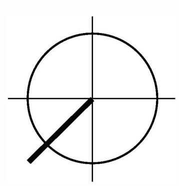


7. Consider angles with a reference angle of 45°. Earlier, you used a 45-45-90 triangle to discover that the tangent of 45° is ____. Use "tangent is slope" to identify each ratio.

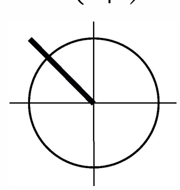
a. $\tan(45°)$



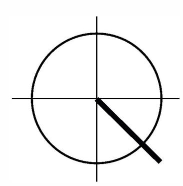
b. $\tan(225°)$



c. $\tan\left(\frac{3\pi}{4}\right)$



d. $\tan(315°)$



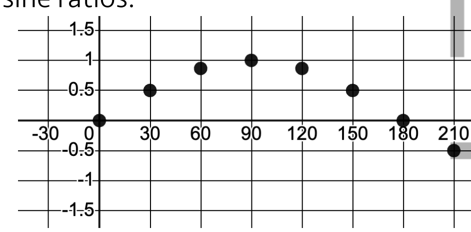
Answer Key

1.	a. 60° or 300° b. 210° or 330°
2.	a. $\tan(45°) = \frac{x}{x} = 1$ b. $\tan(60°) = \frac{x\sqrt{3}}{x} = \sqrt{3}$ c. $\tan(30°) = \frac{x}{x\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
3.	a. $\frac{0}{n}$ is 0 b. $\frac{n}{0}$ is undefined
4.	a. $\cos(\theta) = \frac{A}{1} = A$ $\sin(\theta) = \frac{B}{1} = B$ b. $\tan(\theta) = \frac{B}{A}$ c. $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$
5.	a. $\frac{\text{rise}}{\text{run}} \rightarrow \frac{B}{A}$ b. Since the slope is $\frac{B}{A}$ and $\tan(\theta) = \frac{B}{A}$, an angle's tangent ratio is the slope of the line slanted at that angle.
6.	horizontal slope: 0 vertical slope: undefined a. 0 b. 0 c. undefined d. undefined
7.	The tangent of 45° is 1. a. 1 b. 1 c. −1 d. −1
8.	a. 60° b. $\sqrt{3}$ c. $\tan(60°)$
9.	a. $\sqrt{3}$ b. $-\sqrt{3}$ c. $\sqrt{3}$ d. $-\sqrt{3}$
10.	a. $\frac{\sqrt{3}}{3}$ b. $-\frac{\sqrt{3}}{3}$ c. $\frac{\sqrt{3}}{3}$ d. $-\frac{\sqrt{3}}{3}$
11.	a. $\frac{3}{3} \rightarrow 1$ b. $\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} \rightarrow \frac{\sqrt{3}}{2} \cdot \frac{2}{1} \rightarrow \sqrt{3}$ c. $\tan(180°) = 0$ because the terminal side of an angle at 180° has a slope of 0.
12.	a. undefined b. $\frac{\sqrt{3}}{3}$ c. −1 d. $\sqrt{3}$
13.	a. 135° or 315° b. 60° or 240°
14.	a. cosecant b. secant c. cotangent d. cosine e. tangent f. sine
15.	a. 2 b. $-\frac{4}{3}$ c. −1 d. 1 e. $\frac{1}{0} \rightarrow$ undefined
16.	a. $\frac{3}{2}$ b. $-\frac{5}{4}$ c. $\frac{3}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{3}$ d. $-\frac{1}{4}$
17.	a. $\sin\theta$ b. $\tan\theta$ c. $\cos\theta$ d. $\cot\theta$ e. $\csc\theta$ f. $\sec\theta$
18.	a. $\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ b. $-\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

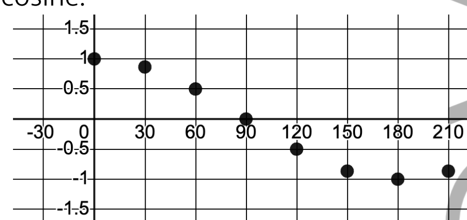
19.	a. $\frac{1}{0} \rightarrow$ undefined b. $\cos\theta = \frac{0}{1} \rightarrow 0$
20.	a. $\sin(30°) = \frac{1}{2} \rightarrow \csc(30°) = 2$ b. $\cos(45°) = \frac{\sqrt{2}}{2} \rightarrow \sec(45°) = \frac{2}{\sqrt{2}} \rightarrow \sqrt{2}$ c. $\tan(60°) = \sqrt{3} \rightarrow \cot(60°) = \frac{\sqrt{3}}{3}$ d. $\cos(90°) = 0 \rightarrow \sec\left(\frac{\pi}{2}\right)$ is undefined
21.	a. $\frac{\pi}{4}$ is 45°; $\frac{2}{\sqrt{2}} \rightarrow \sqrt{2}$ b. 180°; $\frac{1}{0} \rightarrow$ undefined c. 60°; $\frac{2\sqrt{3}}{3}$
22.	a. 135° b. 270°
23.	a. $240° \rightarrow \frac{4\pi}{3}$ b. $315° \rightarrow \frac{7\pi}{4}$
24.	a. $-\frac{1}{2}$ b. $-\frac{\sqrt{2}}{2}$
25.	a. $\cos(\theta) = -\frac{\sqrt{3}}{2}$ $\theta = \frac{5\pi}{6}$ b. $\sin(\theta) = -\frac{\sqrt{2}}{2}$ $\theta = \frac{7\pi}{4}$

26.

a. sine ratios:



b. cosine:



27.	a. $\tan(\theta) = \frac{3}{6} = \frac{1}{2}$ b. $\theta = \tan^{-1}\left(\frac{1}{2}\right) = 26.6°$
28.	a. $\tan(\theta) = -\frac{3}{6} = -\frac{1}{2}$ b. $\tan(\theta) = \frac{-4}{-3} = \frac{4}{3}$
29.	a. $\tan^{-1}\left(-\frac{1}{2}\right) = -26.6°$ The reference ∠ is 26.6°, in Quadrant 4. $\rightarrow \theta = 360° - 26.6° = 334.4°$

CONTENTS

Unit 10

Trigonometric Equations & Identities

SAMPLE PAGES

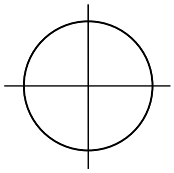
Section 1

Solving Trigonometric Equations

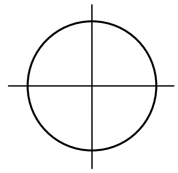
Use this page for taking notes or anything else that helps you learn.

8. Two angles between 0 and 2π have a sine ratio of $\frac{\sqrt{2}}{2}$. On the Unit Circle, there are usually 2 angles with the same trig ratio. Using radians, which angles on the Unit Circle have each sine ratio shown? To help you picture this, mark the 2 angles on the circle.

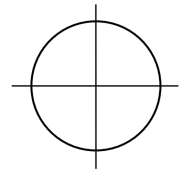
a. $\frac{1}{2}$



b. 0

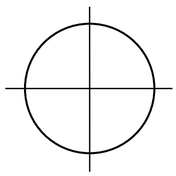


c. $-\frac{\sqrt{3}}{2}$

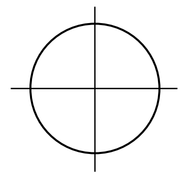


9. The equation $\sin\theta = \frac{1}{2}$ has two solutions: $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$. Which angles on the Unit Circle are solutions to each equation shown, in degrees? Mark the angles on the circle.

a. $\cos\theta = 0$



b. $\tan\theta = 1$



10. When you "solve" an equation, you find all values that make the equation true. Trig equations are not the only ones that can have more than one solution. Solve each equation shown.

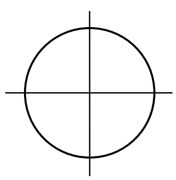
a. $x^2 = 9$

b. $|x| = 9$

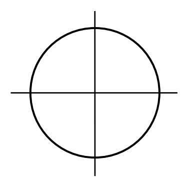
c. $\sqrt{x} = 9$

11. Solve each equation, using degrees over domain $0 \le \theta < 360°$. Mark the angles on the circle.

a. $\cos\theta = -\frac{\sqrt{2}}{2}$



b. $\sin\theta = -\frac{1}{2}$



12. Write and solve the sine equations that only have one solution in the interval $[0, 360°)$.

Simple Trig Equations: Part 2

The next scenarios guide you through trig equations that are more complex. To solve each equation, isolate the trig function and then identify all angles that make the equation true.

13. Solve each equation, using radians. Restrict angles to the following domain: $0 \le \theta < 2\pi$.

a. $\sin\theta + 2 = 2$

b. $3\tan\theta = -3$

14. Solve each equation. Restrict angles to the following domain: $0 \leq \theta < 2\pi$.

a. $\csc\theta = \frac{2}{\sqrt{3}}$

b. $\tan\theta = \pm\sqrt{3}$

15. Solve each equation. Restrict angles to the domain $[0, 2\pi)$.

a. $(\sec\theta)^2 = 4$

b. $2\tan\theta - 7 = -5$

16. The previous equations have had a restricted domain, but angles can be larger than 2π. Using only positive angles, in radians, state the first 4 solutions to each equation below.

a. $\cos\theta = -1$

b. $(\sin\theta)^2 = 1$

c. $1 + \cot\theta = \sqrt{3} + 1$

Advanced Trig Equations: Part 1

17. Now that you have learned how to solve simpler trig equations, consider a more challenging one:

$\sin^2\theta - \sin\theta = 0$ (Do not try to solve the equation.)

Solving this equation combines two concepts: trig equations and factoring. Before you use factoring to solve trig equations, it will help to review some algebra concepts. Simplify each expression.

a. $3\tan\theta + \tan\theta$

b. $(\sin\theta)(\sin\theta)$

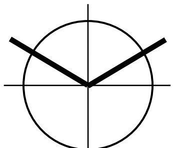
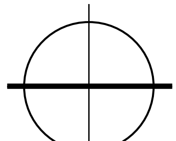
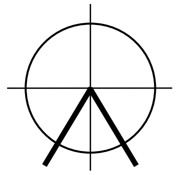
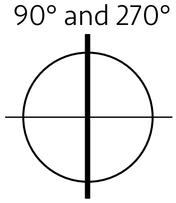
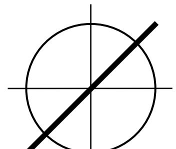
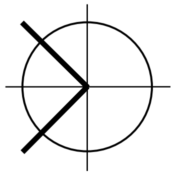
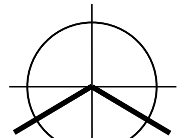
18. Consider the expression $(\sin\theta)^2$. Without parentheses, $\sin\theta^2$, it looks like the angle is squared instead of $\sin\theta$. To avoid confusion, the expression $(\sin\theta)^2$ is written without parentheses as $\sin^2\theta$.

a. The expression $(\cos\theta)^2$ can also be written as ____________.

b. The expression $\tan^3\theta$ can also be written as ____________.

c. The expression $3(\sin\theta)^2$ can also be written as ____________.

Answer Key

1.	a. 45° → tangent ratio is ± 1 b. 30° → tangent ratio is $\pm\frac{\sqrt{3}}{3}$ c. 60° → tangent ratio is $\pm\sqrt{3}$
2.	a. $\frac{1}{2}$ b. $-\frac{\sqrt{2}}{2}$ c. -1 d. $\frac{2}{\sqrt{3}} \rightarrow \frac{2\sqrt{3}}{3}$ e. $\frac{\sqrt{3}}{3}$ f. $\frac{1}{0} \rightarrow$ undefined
3.	$0° \rightarrow (1, 0)$ $45° \rightarrow \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ $90° \rightarrow (0, 1)$ $30° \rightarrow \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ $60° \rightarrow \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ a. $-\frac{\sqrt{3}}{2}$ b. $\frac{1}{0} \rightarrow$ undefined c. -2
4.	45° and 225°
5.	a. $5 \cdot 30° \rightarrow \frac{5\pi}{6}$ b. $7 \cdot 45° \rightarrow \frac{7\pi}{4}$ c. $4 \cdot 60° \rightarrow \frac{4\pi}{3}$
6.	a. $-\frac{\sqrt{3}}{3}$ b. -2
7.	Both ratios are $\frac{\sqrt{2}}{2}$.
8.	a. $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ b. 0 and π c. $\frac{4\pi}{3}$ and $\frac{5\pi}{3}$ a.  b.  c. 
9.	a. 90° and 270°  b. 45° and 225° 
10.	a. $x = \pm 3$ b. $x = \pm 9$ c. only 1 solution: $x = 81$
11.	a. 135° and 225°  b. 210° and 330° 
12.	$\sin\theta = 1 \rightarrow \theta = \frac{\pi}{2}$ $\sin\theta = -1 \rightarrow \theta = \frac{3\pi}{2}$

13.	a. $\sin\theta = 0 \rightarrow \theta = 0, \pi$ b. $\tan\theta = -1 \rightarrow \theta = \frac{3\pi}{4}, \frac{7\pi}{4}$
14.	a. $\sin\theta = \frac{\sqrt{3}}{2} \rightarrow \theta = \frac{\pi}{3}, \frac{2\pi}{3}$ b. $\tan\theta = \sqrt{3} \rightarrow \theta = \frac{\pi}{3}, \frac{4\pi}{3}$ $\tan\theta = -\sqrt{3} \rightarrow \theta = \frac{2\pi}{3}, \frac{5\pi}{3}$
15.	a. $\sec\theta = \pm 2 \rightarrow \cos\theta = \pm\frac{1}{2}$ $\cos\theta = \frac{1}{2} \rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3}$ $\cos\theta = -\frac{1}{2} \rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ b. $2\tan\theta = 2 \rightarrow \tan\theta = 1 \rightarrow \theta = \frac{\pi}{4}, \frac{5\pi}{4}$
16.	a. $\theta = \pi, 3\pi, 5\pi, 7\pi, \ldots$ b. $\sin\theta = \pm 1$ $\sin\theta = 1 \rightarrow \theta = \frac{\pi}{2}, \frac{5\pi}{2}, \ldots$ $\sin\theta = -1 \rightarrow \theta = \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$ c. $\cot\theta = \sqrt{3} \rightarrow \tan\theta = \frac{\sqrt{3}}{3}$ $\rightarrow \theta = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \frac{19\pi}{6}, \ldots$
17.	a. $4\tan\theta$ b. $(\sin\theta)^2$ or $\sin^2\theta$
18.	a. $\cos^2\theta$ b. $(\tan\theta)^3$ c. $3\sin^2\theta$
19.	a. $\sin^2\theta + 3\sin\theta - 3\sin\theta - 9 \rightarrow \sin^2\theta - 9$ b. $3\tan^2\theta - 5\tan\theta - 2$
20.	a. $5\cos^2\theta - 4\cos\theta$ b. $\sin^2\theta + 6\sin\theta + 9$
21.	a. $x = \pm 1$ b. $x = -5, 2$ c. $(x+7)(x+1) = 0 \rightarrow x = -7, -1$
22.	a. $x(x-1) = 0 \rightarrow x = 0, 1$ b. $3x(x^2 - 9) = 0 \rightarrow 3x(x+3)(x-3) = 0$ $x = 0, 3, -3$
23.	a. $x^2(x-3) - 1(x-3) = 0$ $\rightarrow (x-3)(x^2-1) = 0$ $\rightarrow (x-3)(x+1)(x-1) = 0 \rightarrow x = 3, \pm 1$ b. $x^2(x-5) - 4(x-5) = 0$ $\rightarrow (x-5)(x^2-4) = 0$ $\rightarrow (x-5)(x+2)(x-2) = 0 \rightarrow x = 5, \pm 2$

Section 2

General Solutions of Trigonometric Equations

Use this page for taking notes or anything else that helps you learn.

General Solutions of Trig Equations – Part 1

1. If θ is restricted to $[0, 2\pi)$, the equation below has 1 solution. What is the solution?

   $\cos(\theta) = -1$

2. If x is restricted to $[0, 2\pi)$, the equation below has 2 solutions. What are the solutions?

   $\tan(x) = 1$

3. Simplify each expression.

   a. $\tan(\pi)$ b. $\tan(2\pi)$ c. $\tan(3\pi)$

4. Identify the first 4 positive solutions of the equation $\tan(x) = 0$.

5. Simplify each expression.

   a. $\sin\left(\frac{\pi}{2}\right)$ b. $\sin\left(\frac{5\pi}{2}\right)$ c. $\sin\left(\frac{9\pi}{2}\right)$

6. If angles are not restricted to one rotation around the circle, $[0, 2\pi)$, trig equations have infinitely many solutions. Identify the first 4 positive solutions of the equation $\sin(x) = 1$.

7. Though trig equations have infinite solutions, the solutions follow patterns that can be represented by expressions like the one below. If $n \geq 0$, write the first 3 solutions.

   $x = \frac{\pi}{3} + 2\pi n$

8. Consider the equation $\cos(\theta) = -1$. Its solutions are $\theta = \pi, 3\pi, 5\pi, 7\pi\ldots$ These values follow a pattern that can be represented as: $\theta = \pi + 2\pi n$, where n is 0, 1, 2, 3, etc… This is called the general solution. Represent each solution in a general form.

   a. $\theta = \pi, 5\pi, 9\pi, 11\pi\ldots$ b. $\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}\ldots$ c. $\theta = \frac{\pi}{3}, \frac{7\pi}{3}, \frac{13\pi}{3}, \frac{19\pi}{3}\ldots$

9. Write each solution in a general form.

a. $x = 2, 5, 8, 11\ldots$

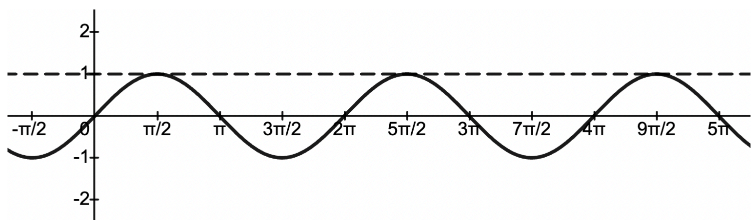
b. $x = \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}\ldots$

c. $x = -2, 4, 10, 16\ldots$

10. Write the general solution of the equation shown.

$\tan(x) = -1$

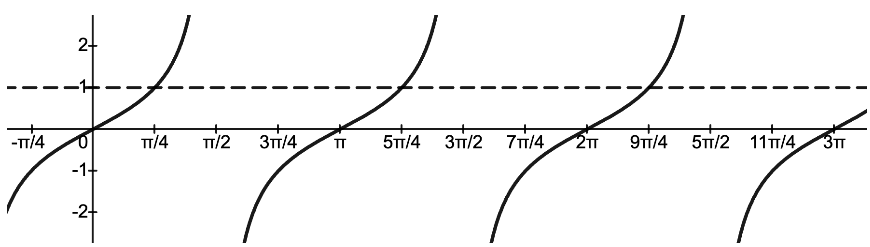
11. These trig equations can also be represented graphically. Consider the graph below. The wave is the graph of the function $y = \sin(x)$. The dashed line is the graph of $y = 1$.



a. Identify the first 4 positive solutions of the equation $\sin(x) = 1$.

b. What is the general solution of the equation $\sin(x) = 1$?

12. Consider the graph below. The repeated S-shaped curves are the graph of the function $y = \tan(x)$. The dashed line is the graph of $y = 1$.



a. Identify the first 4 positive solutions of the equation $\tan(x) = 1$.

b. What is the general solution of the equation $\tan(x) = 1$?

Answer Key

1.	$\theta = \pi$
2.	$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$
3.	$\tan(\pi) = \tan(3\pi) = \tan(5\pi) = 0$
4.	$x = \pi,\ 2\pi,\ 3\pi,\ 4\pi$
5.	$\sin\left(\frac{\pi}{2}\right) = \sin\left(\frac{5\pi}{2}\right) = \sin\left(\frac{9\pi}{2}\right) = 1$
6.	$x = \frac{\pi}{2},\ \frac{5\pi}{2},\ \frac{9\pi}{2},\ \frac{13\pi}{2}$
7.	$x = \frac{\pi}{3}, \frac{7\pi}{3}, \frac{13\pi}{3}$
8.	a. $\theta = \pi + 4\pi n$ b. $\theta = \frac{\pi}{4} + \pi n$ c. $\theta = \frac{\pi}{3} + 2\pi n$
9.	a. $\theta = 2 + 3n$ b. $\theta = \frac{1}{4} + \frac{\pi}{2}n$ c. $\theta = -2 + 6n$
10.	$x = \frac{3\pi}{4} + \pi n$
11.	a. $x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}$ b. $x = \frac{\pi}{2} + 2\pi n$
12.	a. $x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$ b. $x = \frac{\pi}{4} + \pi n$
13.	$x = \frac{\pi}{3} + 2\pi n$ and $x = \frac{5\pi}{3} + 2\pi n$
14.	a. $x = \frac{\pi}{6} + 2\pi n$ and $x = \frac{5\pi}{6} + 2\pi n$ b. $x = \frac{\pi}{3} + \pi n$
15.	$\sin(x) = \frac{\sqrt{3}}{2}$
16.	a. 90° b. 90° c. 90° d. $\frac{x}{2} = 90°$
17.	a. $3x = 90° \rightarrow x = 30°$ b. $\frac{x}{2} = 90° \rightarrow x = 180°$
18.	a. divide by 2: $x = 45°, 225°, 405°$ b. divide by 3: $x = 30°, 150°, 270°$ c. multiply by 2: $x = 180°, 900°, 1620°$
19.	a. $x = 45° + 180n$ b. $x = 30° + 120n$ c. $\frac{x}{2} = 90° + 360n \rightarrow x = 180° + 720n$
20.	a. $x = \frac{\pi}{4} + \pi n$ b. $3x = \frac{\pi}{2} + 2\pi n \rightarrow x = \frac{\pi}{6} + \frac{2\pi}{3}n$ c. $\frac{x}{2} = \frac{\pi}{2} + 2\pi n \rightarrow x = \pi + 4\pi n$
21.	a. $x = \frac{\pi}{6}$ b. $x = \frac{15k}{2}$
22.	a. $x = \frac{\pi}{12} + \frac{2\pi}{3}n$ b. $x = \pi + 4\pi n$

23.	a. $x = \pi,\ 5\pi,\ 9\pi$ b. $x = \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3}$
24.	$2\theta = \frac{\pi}{4} + \pi n \rightarrow \theta = \frac{\pi}{8} + \frac{\pi}{2}n$
25.	$\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$
26.	a. 180° b. 180° c. $x + 1 = 180 \rightarrow x = 179$ d. $x - 30 = 180 \rightarrow x = 210$
27.	a. $\sin\left(\frac{\pi}{2}\right) = 1$ so $x - \frac{\pi}{4} = \frac{\pi}{2}$ $\rightarrow x = \frac{\pi}{2} + \frac{\pi}{4} \rightarrow x = \frac{2\pi}{4} + \frac{\pi}{4} \rightarrow x = \frac{3\pi}{4}$ b. $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ so $x + \frac{\pi}{6} = \frac{5\pi}{6}$ $\rightarrow x = \frac{5\pi}{6} - \frac{\pi}{6} \rightarrow x = \frac{4\pi}{6} \rightarrow x = \frac{2\pi}{3}$ c. $\tan\left(\frac{3\pi}{4}\right) = -1$ so $x - \frac{\pi}{2} = \frac{3\pi}{4}$ $\rightarrow x = \frac{3\pi}{4} + \frac{\pi}{2} \rightarrow x = \frac{3\pi}{4} + \frac{2\pi}{4} \rightarrow x = \frac{5\pi}{4}$
28.	a. $\cos(\theta) = -\frac{\sqrt{2}}{2}$ if $\theta = \frac{3\pi}{4}$ or $\theta = \frac{5\pi}{4}$ $x + \frac{\pi}{4} = \frac{3\pi}{4}$ or $x + \frac{\pi}{4} = \frac{5\pi}{4}$ $x = \frac{\pi}{2}$ or $x = \pi$ b. $\tan(\theta) = \frac{\sqrt{3}}{3}$ if $\theta = \frac{\pi}{6}$ or $\theta = \frac{7\pi}{6}$ $x - \frac{4\pi}{3} = \frac{\pi}{6}$ or $x - \frac{4\pi}{3} = \frac{7\pi}{6}$ $x = \frac{9\pi}{6} \rightarrow \frac{3\pi}{2}$ or $x = \frac{7\pi}{6} + \frac{8\pi}{6} \rightarrow \frac{5\pi}{2}$ c. $\sin(\theta) = -1$ if $\theta = \frac{3\pi}{2}$ or $\theta = \frac{7\pi}{2}$ $3x + \frac{\pi}{2} = \frac{3\pi}{2}$ or $3x + \frac{\pi}{2} = \frac{7\pi}{2}$ $3x = \pi$ or $3x = 3\pi \rightarrow x = \frac{\pi}{3}$ or $x = \pi$
29.	$2x = \frac{\pi}{3} + 2\pi n$ and $2x = \frac{2\pi}{3} + 2\pi n$ $\rightarrow x = \frac{\pi}{6} + \pi n$ and $x = \frac{\pi}{3} + \pi n$
30.	$\frac{\pi}{6}, \frac{7\pi}{6}$ and $\frac{\pi}{3}, \frac{4\pi}{3}$
31.	$3x = \frac{\pi}{6} + 2\pi n$ and $3x = \frac{5\pi}{6} + 2\pi n$ $\rightarrow x = \frac{\pi}{18} + \frac{2\pi}{3}n$ and $x = \frac{5\pi}{18} + \frac{2\pi}{3}n$
32.	$\frac{\pi}{18}, \frac{13\pi}{18}, \frac{25\pi}{18}$ and $\frac{5\pi}{18}, \frac{17\pi}{18}, \frac{29\pi}{18}$

Section 3

Proving Trigonometric Identities

Use this page for taking notes or anything else that helps you learn.

The equation $\sin^2\theta + \cos^2\theta = 1$ is called an identity, a statement that two expressions are equal. Two simple identities are "$2 + 2 = 4$" and "$\sqrt{9} = 3$." The statement "$\sin^2\theta + \cos^2\theta = 1$" is a trigonometric identity. While you already know simple trig identities, you will now discover more complex identities.

Reciprocal & Quotient Identities

1. Write each expression as a reciprocal of sine, cosine or tangent.

   a. $\csc\theta =$ b. $\sec\theta =$ c. $\cot\theta =$

2. Write tangent and cotangent in terms of sine and cosine.

   a. $\tan\theta =$ b. $\cot\theta =$

3. Simplify each expression by converting everything to $\sin\theta$ or $\cos\theta$.

   a. $\sin\theta\cot\theta$ b. $\sec\theta\cos\theta$ c. $3\tan\theta\cot\theta$

4. Use the reciprocal and quotient identities to simplify each expression.

   a. $\frac{\sin\theta + 1}{\sin\theta}$ b. $\frac{\cos\theta}{\cot\theta}$ c. $\frac{\tan\theta + 3\tan^2\theta}{\tan\theta}$

5. There are many other trig identities. One is shown below.

   a. $\cos\theta\tan\theta = \sin\theta$

   The steps below show why the left side equals the right side. Finish the steps.

   $$\cos\theta \cdot \frac{\sin\theta}{\cos\theta} \rightarrow \frac{\cos\theta}{1} \cdot \frac{\sin\theta}{\cos\theta} \rightarrow$$

   b. To "prove" an identify, change expressions until both sides match. Prove the identity shown by simplifying the left side. The first step is done for you.

   $$\frac{\sin\theta + \cos\theta}{\cos\theta} = \tan\theta + 1$$

   $$\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\cos\theta} \rightarrow$$

6. Consider another identity, shown below.

$$\cos\theta\tan\theta\csc\theta = 1$$

Finish this proof: $\cos\theta \cdot \frac{\sin\theta}{\cos\theta} \cdot \csc\theta \;\; \rightarrow$

7. Prove the following identity by manipulating the equation's left side.

$$\frac{\cos\theta + 1}{\sin\theta} = \cot\theta + \csc\theta$$

8. Prove this identity by manipulating the equation's right side.

$$\frac{\cos\theta + 1}{\cos\theta} = \sec\theta + 1$$

Strategies for Proving Identities

Proving a trig identity is like solving a puzzle. You use identities and algebraic operations to manipulate the 2 sides until they are identical. You can work on one side or both sides.

9. Prove each identify by manipulating the left side.

a. $\frac{1}{\sin\theta\cos\theta} = \csc\theta\sec\theta$

b. $\frac{2}{\csc^2\theta} = 2\sin^2\theta$

10. Prove each identify by manipulating the left side.

a. $2\sec\theta + \tan\theta = \frac{2 + \sin\theta}{\cos\theta}$

b. $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$

Answer Key

1.	a. $\frac{1}{\sin\theta}$ b. $\frac{1}{\cos\theta}$ c. $\frac{1}{\tan\theta}$
2.	a. $\frac{\sin\theta}{\cos\theta}$ b. $\frac{\cos\theta}{\sin\theta}$
3.	a. $\sin\theta\cdot\frac{\cos\theta}{\sin\theta}\to\cos\theta$ b. $\frac{1}{\cos\theta}\cdot\cos\theta\to 1$ c. $3\cdot\frac{\sin\theta}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta}\to 3\cdot\frac{\sin\theta\cos\theta}{\sin\theta\cos\theta}\to 3\cdot 1\to 3$
4.	a. $\frac{\sin\theta}{\sin\theta}+\frac{1}{\sin\theta}\to 1+\csc\theta$ b. $\frac{\cos\theta}{\frac{\cos\theta}{\sin\theta}}\to\cos\theta\div\frac{\cos\theta}{\sin\theta}\to\cos\theta\cdot\frac{\sin\theta}{\cos\theta}$ $\to\frac{\sin\theta\cos\theta}{\cos\theta}\to\sin\theta$ c. $\frac{\tan\theta}{\tan\theta}+\frac{3\tan^2\theta}{\tan\theta}\to 1+3\tan\theta$
5.	a. $\frac{\sin\theta\cos\theta}{\cos\theta}\to\sin\theta$ b. $\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\cos\theta}\to\tan\theta+1$
6.	$\cos\theta\cdot\frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\sin\theta}\to\frac{\sin\theta\cos\theta}{\sin\theta\cos\theta}\to 1$
7.	$\frac{\cos\theta}{\sin\theta}+\frac{1}{\sin\theta}\to\cot\theta+\csc\theta$
8.	$\frac{1}{\cos\theta}+1\to\frac{1}{\cos\theta}+\frac{\cos\theta}{\cos\theta}\to\frac{\cos\theta+1}{\cos\theta}$
9.	a. $\frac{1}{\sin\theta}\cdot\frac{1}{\cos\theta}\to\csc\theta\sec\theta$ b. $\frac{2}{1}\cdot\frac{1}{\csc^2\theta}\to 2\sin^2\theta$
10.	a. $\frac{2}{\cos\theta}+\frac{\sin\theta}{\cos\theta}\to\frac{2+\sin\theta}{\cos\theta}$ b. $(\sin\theta+\cos\theta)(\sin\theta+\cos\theta)$ $\to\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta$ $\to\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta$ $\to 1+2\sin\theta\cos\theta$
11.	a. $\frac{\sec^2 u}{\sec^2 u}-\frac{1}{\sec^2 u}\to 1-\cos^2\theta\to\sin^2\theta$ b. $\frac{(1+\tan\theta)(1-\tan\theta)}{1+\tan\theta}\to 1-\tan\theta$
12.	a. $\frac{\sin\theta(\csc\theta+1)}{\sin\theta(\csc\theta-1)}\to\frac{1+\sin\theta}{1-\sin\theta}$ b. $\frac{\cos\theta(1+\sin\theta)}{\cos^2\theta}\to\frac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta}$ $\to\frac{\cos\theta(1+\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}\to\frac{\cos\theta}{1-\sin\theta}$ c. $\frac{\cos\theta+\cos\theta\sin\theta}{1-\sin^2\theta}\to\frac{\cos\theta+\cos\theta\sin\theta}{\cos^2\theta}$ $\to\frac{\cos\theta}{\cos^2\theta}+\frac{\cos\theta\sin\theta}{\cos^2\theta}\to\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}$ $\to\sec\theta+\tan\theta$

13.	Replace $\sec\theta$ with $\frac{1}{\cos\theta}$ $\frac{1+\frac{1}{\cos\theta}}{1-\frac{1}{\cos\theta}}\to\frac{\frac{\cos\theta}{\cos\theta}+\frac{1}{\cos\theta}}{\frac{\cos\theta}{\cos\theta}-\frac{1}{\cos\theta}}\to\frac{\frac{\cos\theta+1}{\cos\theta}}{\frac{\cos\theta-1}{\cos\theta}}$ $\to\frac{\cos\theta+1}{\cos\theta}\cdot\frac{\cos\theta}{\cos\theta-1}\to\frac{\cos\theta+1}{\cos\theta-1}$
14.	a. $\frac{\sin^2\theta}{\sin\theta\cos\theta}+\frac{\cos^2\theta}{\sin\theta\cos\theta}\to\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}$ $\to\frac{1}{\sin\theta\cos\theta}\to\frac{1}{\sin\theta}\cdot\frac{1}{\cos\theta}\to\csc\theta\sec\theta$ b. $\frac{1}{1+\sin\theta}\cdot\frac{1-\sin\theta}{1-\sin\theta}+\frac{1}{1-\sin\theta}\cdot\frac{1+\sin\theta}{1+\sin\theta}$ $\to\frac{1-\sin\theta}{1-\sin^2\theta}+\frac{1+\sin\theta}{1-\sin^2\theta}$ $\to\frac{2}{1-\sin^2\theta}\to\frac{2}{\cos^2\theta}$
15.	a. $\sin^2\theta=1-\cos^2\theta$ b. $\cos^2\theta=1-\sin^2\theta$
16.	a. $\tan^2\theta+1=\sec^2\theta$ b. $\tan^2\theta=\sec^2\theta-1$ c. $\sec^2\theta-\tan^2\theta=1$
17.	a. $\frac{\sin^2\theta}{\sin^2\theta}+\frac{\cos^2\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta}$ $\to 1+\cot^2\theta=\csc^2\theta$ b. $\cot^2\theta=\csc^2\theta-1$ c. $\csc^2\theta-\cot^2\theta=1$
18.	$\sin^2\theta+\cos^2\theta=1$ $\sin^2\theta=1-\cos^2\theta$ $\cos^2\theta=1-\sin^2\theta$ divide both sides by $\cos^2\theta$: $\tan^2\theta+1=\sec^2\theta$ $\tan^2\theta=\sec^2\theta-1$ $\sec^2\theta-\tan^2\theta=1$ divide both sides by $\sin^2\theta$: $1+\cot^2\theta=\csc^2\theta$ $\cot^2\theta=\csc^2\theta-1$ $\csc^2\theta-\cot^2\theta=1$
19.	a. 1 b. $-\cos^2\theta$ c. 1 d. $-\cot^2\theta$
20.	$\frac{\cos\theta}{\sec\theta}-\frac{\sec\theta}{\sec\theta}\to\frac{\cos\theta}{\frac{1}{\cos\theta}}-1$ $\to\cos^2\theta-1\to-\sin^2\theta$
21.	Replace $\tan^2\theta+1$ with $\sec^2\theta$ $\frac{\sec^2\theta}{\tan\theta\csc^2\theta}\to\frac{1}{\tan\theta}\cdot\frac{\frac{1}{\cos^2\theta}}{\frac{1}{\sin^2\theta}}$ $\to\frac{1}{\tan\theta}\cdot\frac{\sin^2\theta}{\cos^2\theta}\to\frac{1}{\tan\theta}\cdot\tan^2\theta\to\tan\theta$